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1. Biography and Introduction

"Read Euler, read Euler. He is the master of us all."-Pierre-Simon Laplace

The following is a brief biography of Leonhard Euler that will cover some of the most important times and achievements of his life:

When Leonhard Euler was born on April 15, 1707 in Basel, Switzerland, no one expected him to become the mathematician and scientist that he would become. Born of his father Paul Euler (a Protestant clergyman) and his pious mother Marguerite Brucker, it seemed as though his life would be devoted to the Church. However, these plans would change after the world-famous Bernoulli family noticed the gifts and talents of young Leonhard. The Eulers and Bernoullis were family friends, and young Euler's relationship with them provided him with the drive to pursue mathematics.

Euler's abilities were displayed at a very young age. He had a great memory, being able to recall vast amounts of information. As a great mental calculator, he could perform calculations that would require the use of pencil and paper for most others. These skills would help him greatly, especially when his sight started deteriorating. At the age of fourteen, he began attending the University of Basel. There he met Johann Bernoulli, who was arguably the greatest mathematician of the time (with the retirement of Newton from mathematics and death of Leibniz). Johann became young Euler's mentor and was greatly impressed by his abilities. This was shocking as Johann had a reputation of being arrogant and would often degrade other people's works.

At the University of Basel, Euler's studies went beyond mathematics. He wrote on law and even earned a master's degree in philosophy. After completing his masters, he attended divinity school to become a member of the clergy. At this point, it seemed as though Euler would be following in his father's footsteps. However, his love of mathematics compelled him to drop his religious studies. From then on, he realized that his true calling was to be a great mathematician.

In 1725, Johann Bernoulli's son Daniel had arrived at the St. Petersburg Academy to study mathematics. The following year, Daniel invited Euler to work at the St. Petersburg Academy. However, the only opening at the time was in the department of medicine and physiology. Although he was not a doctor of any means, Euler agreed to go. Upon arriving in 1727, Euler (luckily) found out that his post was changed to physics. In 1733, Daniel left for Basel as he could not handle the hostility he received in St. Petersburg. His leaving opened up the position of chair of mathematics, which Euler quickly filled.

After getting comfortable in his new, powerful position, Euler decided to tend to his personal life. He married Katharina Gsell and bore 13 children. However, only five of these children lived to be teenagers and of those five, only three of them outlived both Leonhard and Katharina. They remained happily married until the death of Katharina over 40 years later. Three years after her death, Euler would go on to marry Katharina's half sister, which lasted until Euler's death.

Euler's first gained notoriety in the mathematical community in 1735 when he solved the Basel Problem. Pietro Mengoli originally posed this problem in 1644, and no mathematician could solve it for nearly a century. Not even the great Bernoulli family could solve this daunting problem. A discussion of the Basel Problem and Euler's solution can be found in the "Analysis" section. From this point, Euler started producing an extraordinary number of papers over many areas of mathematics.

In the next few years, Euler had to deal with many hardships. There was much political strife after the death of Catherine I. Her death left a hole in the society, leading to heightened xenophobia and civil unrest. This put a strain on Euler and many of the other professors at the St. Petersburg Academy as most of them were not indigenously Russian (the reason why Daniel Bernoulli left). Another "hinderance" to his studies came about at around 1738, when Euler's vision started to deteriorate. Even though Euler believed his eyesight problems stemmed from overwork (especially in his study of cartography), it is believed that he suffered an infection that affected his vision.

Even with these problems surrounding him, Euler still was as prolific as ever. It was during this time that he friended Christian Goldbach, sparking his interest in number theory. He would eventually combine his interest in number theory with analysis to develop the field of analytic number theory. Also, he published his groundbreaking scientific text *Mechanica* in which he applied calculus to Newtonian mechanics. This work is considered a "landmark in the history of physics".

With Euler's immense output came international recognition. People from all over Europe were interested in Leonhard. One such person was the Prussian king Frederick the Great, who offered Euler a job at the Berlin Academy. Euler accepted his offer fairly quickly due to the harsh times affecting Russia. So in 1741, Euler and his family left for Germany. It was during this time that he put out two of his most important works: the 1748 text *Introductio in analysin infinitoroum* and the 1755 text *Institutiones calculi differentialis.* He also discovered his famous namesake identity $e^{\pm i\theta} = \cos\theta \pm i \sin\theta$ at around this time. It was during this time that Frederick the Great asked Euler to tutor his niece, the Princess of Anhalt-Dessau. Euler agreed, and he would write letters to her dealing with mathematics and the sciences. These letters would be compiled into a collection called *Letters of Euler on Different Subjects in Natural Philosophy Addressed to a German Princess.* Surprisingly, these over 200 letters became one of Euler's best selling works. The popularity that *Letters to a German Princess* received is a testament to Euler's ability to explain science and mathematics to the average person.

In 1766, Catherine the Great had invited Euler back to Russia and work at the St. Petersburg Academy. By this time, the frenzy in Russia had calmed down. Euler accepted this offer, and he would stay in Russia until the day he died. It was during this stay in Russia that Katharina died.

When Euler died on September 18, 1783 of a hemorrhage, the world had lost one of its most brilliant minds. His works were so numerous that the St. Petersburg Academy continued publishing them posthumously for over 48 years. His contributions and achievements have been everlasting, having provided the mathematical and scientific community with a deeper understanding of the natural world. [1]

The rest of this paper will be devoted to *some* (emphasis on some) of Euler's most important contributions to mathematics, stretching from his impact on the already established areas of number theory and analysis (of both real and complex variables) to his newly formed field of analytic number theory. The aim is to provide the reader with a brief overview of the the works of Euler with the hope that he or she will put further research into this exciting period of mathematical thought.

2. **Number Theory**

2.1. **Introduction.** Most historians agree that Euler's interest in number theory can be traced back to his friend Christian Goldbach, whom he met at the St. Petersburg Academy in 1727. Originally, the newly discovered calculus of Newton and Leibniz fascinated young Euler. Euler first displayed his ability in number theory after Goldbach had posed the following question:

"Is Fermat's observation known to you, that all numbers $2^{2^n}+1$ are prime? He said he could not prove it; nor has anyone else done so to my knowledge."

Euler went on to find a counterexample to this prime number conjecture. He found that $2^{2^5}+1 = 4,294,967,297$ is divisible by 641. After this discovery Euler's passion for number theory escalated. Many volumes of his work are devoted to number theory, including four volumes in his *Opera Omnia*. Even though his works are numerous, we will look at Euler's contribution to number theory in the realm of perfect numbers (with a side note to the amicable numbers).

2.2. **Perfect Numbers and the Euclid-Euler Theorem.** Before we begin to look at Euler's analysis of perfect numbers, we shall define the following:

Definition A whole number is perfect if it is equal to the sum of its proper divisors.

The most basic example of a perfect number is 6 because its proper divisors (1,2, and 3) summed together equal 6 (1+2+3=6). The idea of perfect numbers was known since the time of the Ancient Greeks, with Euclid in particular doing much work in the study of these curious numbers.

Another class of interesting numbers are amicable numbers, which Euler wrote about in his paper *De numeris amicabilibus*. Amicable numbers are two numbers m and n, where the sum of the proper divisors of m is n and vice versa. The first such pair of numbers to occur are 220 and 284. Euler was able to come up with fifty nine pairs, while only three pairs were discovered since the time of the Ancient Greeks.

Euler's contributions to perfect numbers start with the advent of his $\sigma(n)$.

Definition $\sigma(n)$ is the sum of all whole number divisors of n.

An example is $\sigma(6) = 1 + 2 + 3 + 6 = 12$. One can see that the sum of the proper divisors of n is $\sigma(n) - n$. (If m and n are amicable numbers, then $\sigma(m) = m + n = \sigma(n)$). The following are various properties come out of $\sigma(n)$ that deal with prime and perfect numbers:

1. p is prime iff $\sigma(p) = p + 1$
2. N is perfect iff $\sigma(N) = N + N = 2N$
3. If p is prime, $\sigma(p^r) = (p^{r+1} - 1)/(p - 1)$ (If we let $N = 2^r$ then

$$\sigma(N) = \sigma(2^r) = \frac{2^{r+1} - 1}{2 - 1} = 2^{r+1} - 1 = 2(2^r) - 1 = 2N - 1$$

showing that no power of 2 is perfect.)
4. If p and q are different primes, then $\sigma(pq) = \sigma(p)\sigma(q)$
5. If a and b are relatively prime, then $\sigma(ab) = \sigma(a)\sigma(b)$

Using these five properties of $\sigma(n)$, Euler proved the following theorem (originally stated by Euclid) that relates even perfect numbers to primes in the form of $2^k - 1$, which are called Mersenne primes. Euclid's original proof did not have the even restriction.

Theorem 2.1 (Euclid-Euler Theorem). *If N is an even perfect number, then $N = 2^{k-1}(2^k - 1)$, where $2^k - 1$ is prime.*

Proof. First, we assume that N is even and perfect. We can write $N = 2^{k-1}b$ where b is an odd number and $k > 1$ since N is even and thus has 2 as at least one factor. Since N is perfect, we can use property (2) to write $\sigma(N) = 2N = 2(2^{k-1})b = 2^k b$. Since 2^{k-1} and b are relatively prime, properties (3) and (5) state that

$$\sigma(N) = \sigma(2^{k-1}b) = \sigma(2^{k-1})\sigma(b) = (2^k - 1)\sigma(b).$$

Setting these two equations for $\sigma(N)$ equal to each other,

$$2^k b = (2^k - 1)\sigma(b)$$

so that

$$\frac{2^k}{2^k - 1} = \frac{\sigma(b)}{b}.$$

It is apparent that the fraction on the LHS is reduced. However, Euler had to decide whether the RHS was reduced. Setting

$$\sigma(b) = c2^k$$

and

$$b = c(2^k - 1),$$

Euler split up two cases for the value of c.

Case 1. $c > 1$
Euler stated that the divisors of b which are $1, b, c$ and $2^k - 1$ are all different. The following six items show that the divisors are all different:

(a) $1 \neq b$. If that was the case, then $N = 2^{k-1}$, which would not be possible because powers of 2 are not perfect.
(b) $1 \neq c$ since we are assuming $c > 1$.
(c) $1 \neq 2^k - 1$. If that was the case then $2^k = 2$ and therefore $N = 2^{k-1}b = b$. But since b is odd, this would make N odd, which contradicts the fact that N is even.
(d) $b \neq c$. If that was the case then $b = b(2^k - 1) \rightarrow 1 = 2^k - 1$ which leads back to the problem in (c).
(e) $b \neq 2^k - 1$. If that was the case then $b = cb$, which leads to the same problem as (b).
(f) If $c = 2^k - 1$, then $b = c^2$. This implies that b has three different divisors $(1, c, c^2)$. Therefore, $\sigma(b)$ would be at least $1 + c + c^2$. However, $\sigma(b) = c2^k = c[(2^k - 1) + 1] = c[c + 1] = c^2 + c$. This implies that $\sigma(b) = c^2 + c \geq 1 + c + c^2$, which is obviously a fallacy . Therefore, $c \neq 2^k - 1$.

These six contradictions prove that the four divisors of b are different. This implies that

$$\sigma(b) \geq 1 + b + c + 2^k - 1 = b + c + 2^k = c(2^k - 1) + c + 2^k = 2^k(c+1) > c2^k = \sigma(b)$$

and so this case is impossible.

Case 2. $c = 1$
If $c = 1$, then $b = c(2^k - 1) = 2^k - 1$ and so

$$\sigma(b) = c2^k = 2^k = (2^k - 1) + 1 = b + 1.$$

Since $\sigma(b) = b + 1$, we can conclude that b is prime. Therefore, Euler has shown that if N is an even perfect number, then $N = 2^k - 1(2^k - 1)$, where $2^k - 1$ is a prime number. □

This concludes Euler's contribution on the field of number theory. We have barely skimmed the surface of Euler's impact in this field. One of the most important proofs Euler devised in the field of number theory was his proof of Fermat's little theorem, which he generalized into his namesake Euler's theorem. Although he never proved it, Euler conjectured the law of quadratic reciprocity (it would be proven by the great mathematician Carl Gauss) [2]. With this said, if one would consider Euler as just a number theorist, he would still be ranked as one of the world's greatest mathematicians. We will now change gears and investigate Euler's huge impact on the field of analysis, especially in the study of infinite series. [1]

3. **Analysis**

3.1. **Introduction.** This section is devoted to Euler's work in the field of analysis, with reference to his work on the logarithm and exponential functions, as well as his work with infinite series. His work on logarithms and exponentials is found on chapters VI and VII of the *Introductio*. Euler's findings in this area of mathematics would be considered some of the greatest achievements in modern analysis, especially with the discovery of the natural logarithm and the natural base e, which is referred to as Euler's number. Another important discovery was that of another pivotal constant, denoted as γ, which is aptly called Euler's constant.

3.2. **Exponential Power Series and Euler's Number.** Euler first devised an infinite series expansion for the function $y = a^x$ for $a > 1$. He started by letting ω be some infinitesimal number so that $a^\omega = 1+\psi$, where ψ is another infinitesimal number. Because both ω and ψ are infinitesimal, Euler related the two by $\psi = k\omega$. This implies that $a^\omega = 1 + k\omega$. Therefore, k depends on a.

From this point, Euler was now ready to find a series expansion for e^x. He let $j = x/\omega$, which results in $a^x = a^{\omega^{(x/\omega)}} = (1 + k\omega)^j = (1 + kx/j)^j$. Using the binomial expansion theorem:

$$a^x = 1 + j\left(\frac{kx}{j}\right) + \frac{j(j-1)}{2\cdot 1}\left(\frac{kx}{j}\right)^2 + \frac{j(j-1)(j-2)}{3\cdot 2\cdot 1}\left(\frac{kx}{j}\right)^3 + \frac{j(j-1)(j-2)(j-3)}{4\cdot 3\cdot 2\cdot 1}\left(\frac{kx}{j}\right)^4 + \cdots$$

$$= 1 + kx + \frac{j-1}{j}\left(\frac{k^2x^2}{2\cdot 1}\right) + \frac{(j-1)(j-2)}{j\cdot j}\left(\frac{k^3x^3}{3\cdot 2\cdot 1}\right) + \frac{(j-1)(j-2)(j-3)}{j\cdot j\cdot j}\left(\frac{k^4x^4}{4\cdot 3\cdot 2\cdot 1}\right) + \cdots.$$

Since x is a finite number and ω is infinitesimal, j is infinitely large. So Euler's logic was that

$$\frac{j-1}{j} = \frac{j-2}{j} = \frac{j-3}{j} = \cdots = \frac{j-n}{j} = 1.$$

As we will see later as well, Euler did not use the notion of a limit in his analyses. Rather, he worked with infinitely big and infinitely little. Anyway, Euler was correct in his assumption. Therefore, the series for a^x followed as:

$$a^x = 1 + kx + \left(\frac{k^2x^2}{2 \cdot 1}\right) + \left(\frac{k^3x^3}{3 \cdot 2 \cdot 1}\right) + \left(\frac{k^4x^4}{4 \cdot 3 \cdot 2 \cdot 1}\right) + \cdots .$$

Immediately letting x=1:

$$a = 1 + k + \left(\frac{k^2}{2 \cdot 1}\right) + \left(\frac{k^3}{3 \cdot 2 \cdot 1}\right) + \left(\frac{k^4}{4 \cdot 3 \cdot 2 \cdot 1}\right) + \cdots .$$

Euler then arbitrarily defined the constant a as the base when k=1. This results in

$$a = 1 + 1 + \left(\frac{1}{2 \cdot 1}\right) + \left(\frac{1}{3 \cdot 2 \cdot 1}\right) + \left(\frac{1}{4 \cdot 3 \cdot 2 \cdot 1}\right) + \cdots$$

which he calculated to be 2.7182818...He then designated this number as the letter "e". Therefore he concluded that

$$e^x = 1 + x + \frac{x^2}{2 \cdot 1} + \frac{x^3}{3 \cdot 2 \cdot 1} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

3.3. **Logarithm Power Series.** Now that Euler had discovered an elegant power series representation for the exponential function, his next logical step was to find a representation for the logarithm function. First, Euler referred back to his infinitesimal ω. Knowing that $e^{\omega} = 1 + \omega$ for such a small value of ω, it immediately followed that $\omega = \log(1 + \omega)$ which implies that $j\omega = j\log(1 + \omega) = \log(1 + \omega)^j$. Since ω is positive, (albeit very small), $(1 + \omega)^j > 1$. Euler then stated that $x = (1 + \omega)^j - 1$ for some positive x. This results lead Euler to three important realizations:

1. $\omega = (1 + \omega)^{1/j} - 1)$
2. $1 + x = (1 + \omega)^j = e^{\omega j} \rightarrow \log(1 + x) = j\omega$
3. $j- > \infty.$

j goes to infinity because $\log(1 + x)$ is finite and ω is infinitesimal. Now, Euler utilizes the binomial expansion theorem (again):

$$\log(1+x) = j\omega = j[(1+x)^{1/j} - 1] = j\left[1 + \left(\frac{1}{j}\right)x + \frac{\left(\frac{1}{j}\right)\left(\frac{1}{j} - 1\right)}{2 \cdot 1}x^2 + \frac{\left(\frac{1}{j}\right)\left(\frac{1}{j} - 1\right)\left(\frac{1}{j} - 2\right)}{3 \cdot 2 \cdot 1}x^3 + \cdots\right] - j$$

$$= x - \frac{j-1}{2j}x^2 + \frac{(j-1)(2j-1)}{2j \cdot 3j}x^3 + \frac{(j-1)(2j-1)(3j-1)}{2j \cdot 3j \cdot 4j}x^4 + \cdots .$$

Euler then used the fact that since j is infinitely large, $\frac{(j-1)}{2j} = \frac{1}{2}$ and so forth. Therefore we have that

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots .$$

Euler now wanted to use this expansion as a tool to calculate logarithms. Because of the radius of convergence of this series, Euler was not able to directly calculate logarithms on a grand scale. However, through manipulation he found an ingenious method. First, he replaced x by -x in the series and subtracted the two.

$$\log(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\log(1+x)-\log(1-x) = \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right] - \left[-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right] = 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \cdots$$

$$\log\frac{1+x}{1-x} = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right].$$

The behavior of this series is much better than that of the original logarithm expansion and it allows for the computation of many logarithms. As an example, if $x = \frac{1}{3}$ then

$$\log\frac{1+\frac{1}{3}}{1-\frac{1}{3}} = \log 2 = 2\left[\frac{1}{3} + \frac{1}{81} + \frac{1}{15309} + \cdots\right] = 0.693135\ldots$$

We will now show how Euler determined that the derivative of the logarithm function is $\frac{1}{x}$ through the use of the expansion. Starting with $y = \log x$, its differential form is $dy = \log(x + dx) - \log x$. From here, Euler used logarithm rules and the series expansion to come up with:

$$dy = \log(x+dx) - \log x = \log\left(\frac{x+dx}{x}\right) = \log\left(1 + \frac{dx}{x}\right) = \frac{dx}{x} - \frac{(dx/x)^2}{2} + \frac{(dx/x)^3}{3} - \frac{(dx/x)^4}{4} + \cdots.$$

Euler (correctly) deduced that he could drop off the higher power dx/x terms since dx itself is infinitely small. Therefore, he stated that

$$dy = \frac{dx}{x} \to \frac{dy}{dx} = \frac{d}{dx}\log x = \frac{1}{x}.$$

3.3.1. *Euler's constant* γ. We will now end the discussion on Euler's contribution to the study of logarithms with the vastly important mathematical constant γ. Arising from an intimate relationship between the harmonic series and the logarithm function, γ appears throughout many areas of mathematics, including analysis and number theory. It has not been proven whether γ is irrational or transcendental, but most mathematicians believe it to be both.

Theorem 3.1. *The harmonic series* $\left(\sum_{k=1}^{\infty} \frac{1}{k}\right)$ *diverges*

Proof. Euler showed the divergence of the harmonic series by starting with the logarithm series expansion with $-x$ in place of x:

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots.$$

Then he let $x = 1$ into this series so that

$$\log 0 = -\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots\right).$$

Concluding,

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = -\log 0 = \log\left(\frac{1}{0}\right) = \log\infty = \infty.$$

□

This is not the most rigorous proof that there is for the divergence of the harmonic series. However, Euler went further with this connection. Substituting $x = 1/n$ into the logarithm series expansion:

$$\log\left(1 + \frac{1}{n}\right) = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots.$$

Therefore

$$\frac{1}{n} = \log\left(\frac{n+1}{n}\right) + \frac{1}{2n^2} - \frac{1}{3n^3} + \cdots .$$

As n gets large, $1/n$ approaches $\log[(n+1)/n]$. Euler then realized that summing the harmonic series would turn into summing logarithms. Armed with this intuition, he then made rows with $n = 1, 2, 3, \cdots$ to get

$$1 = \log 2 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots$$
$$\frac{1}{2} = \log\left(\frac{3}{2}\right) + \frac{1}{8} - \frac{1}{24} + \frac{1}{64} - \cdots$$
$$\frac{1}{3} = \log\left(\frac{4}{3}\right) + \frac{1}{18} - \frac{1}{81} + \frac{1}{324} - \cdots$$
$$\frac{1}{n} = \log\left(\frac{n+1}{n}\right) + \frac{1}{2n^2} - \frac{1}{3n^3} + \cdots .$$

Summing the columns:

$$\sum_{k=1}^{n} \frac{1}{k} = \left[\log 2 + \log\left(\frac{3}{2}\right) + \log\left(\frac{4}{3}\right) + \cdots + \log\left(\frac{n+1}{n}\right)\right] + \frac{1}{2}\left[1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2}\right]$$
$$-\frac{1}{3}\left[1 + \frac{1}{8} + \frac{1}{27} + \cdots + \frac{1}{n^3}\right] + \frac{1}{4}\left[1 + \frac{1}{16} + \frac{1}{81} + \cdots + \frac{1}{n^2}\right] - \cdots .$$

Euler then simplified the logarithms using logarithm rules:

$$\left[\log 2 + \log\left(\frac{3}{2}\right) + \log\left(\frac{4}{3}\right) + \cdots + \log\left(\frac{n+1}{n}\right)\right] = \log\left(2 \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{n+1}{n}\right) = \log(n+1).$$

Euler (the great calculator that he was) then estimated the numerical terms and came to the expression

$$\sum_{k=1}^{n} \frac{1}{k} \approx \log(n+1) + 0.577218.$$

Subtracting the logarithm to the other side and letting n go to infinity

$$\gamma = \lim_{n\to\infty}\left[\sum_{k=1}^{n} \frac{1}{k} - \log(n+1)\right].$$

Nowadays, we rewrite the formula for γ as

$$\gamma = \lim_{n\to\infty}\left[\sum_{k=1}^{n} \frac{1}{k} - \log(n)\right].$$

A quick proof of this is the following:

$$\lim_{n\to\infty}\left[\sum_{k=1}^{n} \frac{1}{k} - \log(n)\right] = \lim_{n\to\infty}\left[\sum_{k=1}^{n} \frac{1}{k} - \log(n+1) + \log(n+1) - \log(n)\right]$$
$$= \lim_{n\to\infty}\left[\sum_{k=1}^{n} \frac{1}{k} - \log(n+1)\right] + \lim_{n\to\infty} \log\left(1 + \frac{1}{n}\right) = \gamma + \log 1 = \gamma + 0 = \gamma.$$

As a slight aside, some other interesting ways γ appears is in the following three formulas:

$$\gamma = -\int_0^\infty e^{-x} \log x \, dx$$

$$\gamma = \left[\frac{1}{2 \cdot 2!} - \frac{1}{4 \cdot 4!} + \frac{1}{6 \cdot 6!} - \cdots\right] - \int_1^\infty \frac{\cos x}{x} \, dx$$

$$\gamma = \lim_{x \to 1^+} \sum_{n=1}^\infty \left(\frac{1}{n^x} - \frac{1}{x^n}\right)$$

3.4. **Basel Problem.** The Basel Problem is considered by many to be Euler's first huge breakthrough in mainstream mathematics. As stated in the introduction, the solution had been eluding mathematicians since the problem was proposed in the 1600s. With that being said, it only took the mind of a twenty four year old Euler to brilliantly solve the problem.

Theorem 3.2 (Basel Problem)**.**

$$\sum_{k=1}^\infty \frac{1}{k^2} = \frac{\pi^2}{6}$$

Proof. Euler first created the polynomial

$$P(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

where $P(x)$ is an infinite series. Next, Euler used some of his famous trickery to determine the roots of this polynomial. So for $x \neq 0$:

$$P(x) = x\left[\frac{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots}{x}\right] = \frac{x - \frac{x^3}{3!} + \frac{x^4}{5!} - \frac{x^7}{7!} + \cdots}{x} = \frac{\sin x}{x}$$

The roots of $P(x)$ occur then when $\sin x = 0$ i.e. when $x = \pm k\pi$ i.e. $k \in \mathbb{Z}^+$ Knowing this, Euler factored:

$$P(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

$$= \left(1 - \frac{x}{\pi}\right)\left(1 - \frac{x}{-\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 - \frac{x}{-2\pi}\right)\cdots = \left[1 - \frac{x^2}{\pi^2}\right]\left[1 - \frac{x^2}{4\pi^2}\right]\left[1 - \frac{x^2}{9\pi^2}\right]\cdots$$

Euler expanded the RHS to yield

$$1 - \left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \cdots\right)x^2 + \cdots$$

where the $+\cdots$ term represents higher (and unknown) even powers of x. Euler then equated the x^2 coefficients to the original form of $P(x)$ to get:

$$1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots = 1 - \left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \cdots\right)x^2 + \cdots$$

$$-\frac{1}{3!} = -\left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \cdots\right) = -\frac{1}{\pi^2}\left(1 + \frac{1}{4} + \frac{1}{9} + \cdots\right)$$

Therefore, Euler determined that

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots = \sum_{k=1}^\infty \frac{1}{k^2} = \frac{\pi^2}{6}$$

□

The astute reader will realize that this solution is subject to attacks on its rigor, as it makes use of assumptions and disregards possible terms. Euler even started questioning his own argument. However, Euler had great confidence in his solution, and a calculation of $\pi^2/6$ showed that it was identical to previous numerical estimates. Later on, Euler had produced alternate proofs of the Basel problem.

An interesting corollary to this theorem arises if one turns to the Wallis Formula. In 1655, British mathematician John Wallis (obviously independent of Euler) determined a very beautiful formula relating odd and even numbers intimately with π.

Corollary 3.3 (Wallis Formula)**.**

$$\frac{2}{\pi} = \frac{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdots}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdots}$$

Proof. First, Euler plugged in $x = \pi/2$ into $P(x)$. Using the infinite product form of $P(x)$,

$$P\left(\frac{\pi}{2}\right) = \left[1 - \frac{(\pi/2)^2}{\pi^2}\right]\left[1 - \frac{(\pi/2)^2}{4\pi^2}\right]\left[1 - \frac{(\pi/2)^2}{9\pi^2}\right].$$

Therefore:

$$\frac{\sin(\pi/2)}{\pi/2} = \left[1 - \frac{1}{4}\right]\left[1 - \frac{1}{16}\right]\left[1 - \frac{1}{36}\right]\cdots = \frac{3}{4} \cdot \frac{15}{16} \cdot \frac{35}{36} \cdots$$

Factoring the numerator and denominator,

$$\frac{2}{\pi} = \frac{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdots}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdots}$$

[1] □

A good bridge between this section and the next one of analytic number theory is to talk about the Riemann zeta function. The Riemann zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

It is currently known (and proven by Euler) that if s is even (i.e. $s = 2k$) then

$$\zeta(2k) = \frac{(-1)^{k-1} B_{2k} (2\pi)^{2k}}{2(2k)!}$$

where B_{2k} is a Bernoulli number defined by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n.$$

The first few Bernoulli numbers are $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, and $B_4 = -1/30$. There are no odd B_n except for B_1. As one can check, $\zeta(2) = \pi^2/6$. Finding $\zeta(2k+1)$ for $k \geq 1$ is still unknown and is currently a hot topic in mathematics.

The Riemann zeta function comes up a lot in analysis and number theory, and there is much research in the connection between the Zeta function and prime number theory. Arguably the most famous modern conjecture in mathematics, the Riemann hypothesis may hold some important secrets. The hypothesis, proposed by Riemann himself, states that the real part of any non-trivial zero of the zeta function is 1/2. If the Riemann hypothesis is proven to be true, one can immediately prove the prime number theorem. As we'll see in the next section, we can thank Euler for this intimate relationship between analysis and number theory. [2]

4. **Analytic Number Theory**

4.1. **Introduction.** The origins of Euler's analytic number theory stem from observations that were recorded in his important 1737 paper *Variae observationes circa series infinitas.* As the title suggests, this paper was on various observations about infinite series. The initial investigations in this paper wouldn't lead one to believe that there is some relationship to number theory within the text. However, Euler had proven essential theorems in number theory through the use of infinite series analysis in this paper. The following will be a short summary of Euler's most interesting and important discoveries within this new field.

4.2. **Harmonic Series and the Infinitude of Primes.** Euler had known that the harmonic series was divergent. With this in mind, he was able to manipulate this series to express is as a beautiful product that includes every prime.

Theorem 4.1.

$$\sum_{k=1}^{\infty} \frac{1}{k} = \prod_p \frac{1}{1-\frac{1}{p}} = \frac{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdots}{1 \cdot 2 \cdot 4 \cdot 6 \cdot 10 \cdots}$$

Proof. Euler first set x equal to the harmonic series. Although x is infinite (since the harmonic series diverges), Euler subtracted $x/2$ from x to get

$$\frac{1}{2}x = x - \frac{1}{2}x = \left[1 + \frac{1}{2} + \frac{1}{3} + \cdots\right] - \left[\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots\right] = 1 + \frac{1}{3} + \frac{1}{5} + \cdots .$$

Dividing through by 1/3,

$$\frac{1}{6}x = \frac{1}{3}\left[1 + \frac{1}{3} + \frac{1}{5} + \cdots\right] = \frac{1}{3} + \frac{1}{9} + \frac{1}{15} + \cdots .$$

Subtracting this again from $\frac{1}{2}x$

$$\frac{1}{2}x - \frac{1}{6}x = \left[1 + \frac{1}{3} + \frac{1}{5} + \cdots\right] - \left[\frac{1}{3} + \frac{1}{9} + \frac{1}{15} + \cdots\right],$$

so that

$$\frac{1 \cdot 2}{2 \cdot 3}x = 1 + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots .$$

Euler began to notice a pattern developing, so he continued this subtraction process, which gave

$$\frac{1 \cdot 2}{2 \cdot 3}x - \frac{1}{5}\left[\frac{1 \cdot 2}{2 \cdot 3}x\right] = \left[1 + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots\right] - \left[\frac{1}{5} + \frac{1}{25} + \frac{1}{35} + \frac{1}{55} + \cdots\right]$$

and therefore

$$\frac{1 \cdot 2 \cdot 4}{2 \cdot 3 \cdot 5}x = 1 + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \cdots .$$

Euler came to the conclusion that if this method of dividing and subtracting were to continue infinitely, then

$$\frac{1 \cdot 2 \cdot 4 \cdot 6 \cdot 10 \cdots}{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdots}x = 1$$

so that after solving for x,

$$\sum_{k=1}^{\infty} \frac{1}{k} = \frac{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdots}{1 \cdot 2 \cdot 4 \cdot 6 \cdot 10 \cdots} = \prod_{p} \frac{1}{1 - \frac{1}{p}}.$$

□

Although this is a very hairy "proof" by today's standards, it turns out that Euler's analysis does have validity. It was shown by Leopold Kronecker in 1876 that

$$\sum_{k=1}^{\infty} \frac{1}{k^s} = \prod_{p} \frac{1}{1 - \frac{1}{p^s}}$$

for $s > 1$. Euler's result comes if we let $s \to 1^+$. A very interesting corollary comes out of this theorem.

Corollary 4.2. *There are an infinitude of primes.*

Proof. We know that $\sum_{k=1}^{\infty} \frac{1}{k} \to \infty$. Therefore, $\prod_p \frac{1}{1-\frac{1}{p^s}}$ also $\to \infty$. In order for a product to be infinitely large, it must have infinitely many factors. This implies that the number of primes is infinite since they are the factors of $\prod_p \frac{1}{1-\frac{1}{p^s}}$. □

We will now go into the final area of Euler's study that will be discussed in this paper: Euler's impact on the field of complex variables. Many of his discoveries and theorems in this field of mathematics have had a lasting impact on both pure mathematics and the sciences. [1]

5. Complex Variables

5.1. **Introduction.** Euler had described $\sqrt{-1}$ as "...neither nothing, nor greater than nothing, nor less than nothing..."

Some of Euler's most famous discoveries are found within the field of complex variables. His namesake formula is one of the most talked about in today's schools due to its simple beauty and applicability. The monumental discoveries made by Euler had laid down the groundwork for future mathematicians like Gauss, Riemann, Weierstrauss and Cauchy to develop the study of complex analysis throughout the 19th century. Euler didn't build up this area of mathematics from the ground up; mathematicians like De Moivre had done some work on developing the study of complex numbers before Euler was even born.

5.2. **From De Moivre's Theorem to Euler's Formula.** Euler's interest in analyzing expressions like $\cos\theta \pm i\sin\theta$ came from the factorization of $\sin^2\theta + \cos^2\theta = 1$ into $(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)$=1. Euler then observed what happens when one multiplies $\cos\theta \pm i\sin\theta$ by $\cos\phi \pm i\sin\phi$:

$$(\cos\theta \pm i\sin\theta)(\cos\phi \pm i\sin\phi) = (\cos\theta\cos\phi - \sin\theta\sin\phi) \pm i(\sin\theta\cos\phi + \cos\theta\sin\phi).$$

By angle addition formulas for cosine and sine, this equals $\cos(\theta + \phi) \pm i\sin(\theta + \phi)$.

Taking $\theta = \phi$:

$$(\cos\theta \pm i\sin\theta)^2 = \cos(2\theta) \pm i\sin(2\theta)$$

After a little thought, Euler extended this argument to higher powers. He then made the statement (what we now call De Moivre's Theorem):

$$(\cos\theta \pm i\sin\theta)^n = \cos(n\theta) \pm i\sin(n\theta)$$

We will now see how Euler applied this discovery of the power series of the sine and cosine functions.

Theorem 5.1.

$$\cos x = 1 - \frac{x^2}{1\cdot 2} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} - \frac{x^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} + ...$$

and

$$\sin x = x - \frac{x^3}{1\cdot 2\cdot 3} + \frac{x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{x^7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} + ...$$

Proof. From Euler's observations previously,

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n \ \textit{and} \ \cos n\theta - i\sin n\theta = (\cos\theta - i\sin\theta)^n (n \geq 1)$$

Adding these together and dividing by 2, Euler determined that

$$\cos n\theta = \frac{(\cos\theta + i\sin\theta)^n + (\cos\theta - i\sin\theta)^n}{2}$$

Using the binomial expansion theorem to expand the powers on the RHS Euler got:

$$\cos n\theta = \frac{1}{2}\left[\cos^n\theta + \frac{ni\cos^{n-1}\theta\sin\theta}{1} - \frac{n(n-1)\cos^{n-2}\theta\sin^2\theta}{1\cdot 2} - \frac{n(n-1)(n-2)i\cos^{n-3}\theta\sin^3\theta}{1\cdot 2\cdot 3} + \cdots\right] +$$

$$\frac{1}{2}\left[\cos^n\theta - \frac{ni\cos^{n-1}\theta\sin\theta}{1} - \frac{n(n-1)\cos^{n-2}\theta\sin^2\theta}{1\cdot 2} + \frac{n(n-1)(n-2)i\cos^{n-3}\theta\sin^3\theta}{1\cdot 2\cdot 3} + \cdots\right]$$

$$= \cos^n\theta - \frac{n(n-1)\cos^{n-2}\theta\sin^2\theta}{1\cdot 2} + \frac{n(n-1)(n-2)(n-3)\cos^{n-4}\theta\sin^4\theta}{1\cdot 2\cdot 3\cdot 4} - \cdots .$$

Now, this is where Euler's characteristic lack of rigor shows up. He let x=$n\theta$, where n is infinitely large. Therefore, $\theta = x/n$ is infinitesimally small. By doing this, he saw that $\cos\theta = 1$ and $\sin\theta = \theta = x/n$. This was Euler's notion of what mathematicians now call the limit i.e.

$$\lim_{\theta\to 0}\cos\theta = 1 \ \textit{and} \ \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

Because n is huge, Euler believed that he could replace the n-1, n-2, n-3 etc. terms by n, which would again be considered non-rigorous by today's standards. This led him to rewrite the cosine series as

$$\cos x = 1^n - \frac{n\cdot n\cdot (1)^{n-2}(x/n)^2}{1\cdot 2} + \frac{n\cdot n\cdot n\cdot n\cdot (1)^{n-4}(x/n)^4}{1\cdot 2\cdot 3\cdot 4} - \cdots$$

$$= 1 - \frac{x^2}{1\cdot 2} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} - \frac{x^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} + \cdots .$$

This is Euler's "proof" of the expansion of $\cos x$. The $\sin x$ power series can be derived in the same manner. □

Now, we will use De Moivre's Theorem and these trigonometric power series to show how Euler proved his namesake theorem in *three* different ways.

Theorem 5.2 (Euler's Formula)**.** *For any real x,* $e^{ix} = \cos x + i\sin x$

Proof 1. Euler started with the DeMoivre's theorem result that

$$\cos n\theta = \frac{(\cos\theta + i\sin\theta)^n + (\cos\theta - i\sin\theta)^n}{2}$$

and

$$\sin n\theta = \frac{(\cos\theta + i\sin\theta)^n - (\cos\theta - i\sin\theta)^n}{2i}.$$

In a similar fashion as before, he let n be an infinitely big number so that $\theta = x/n$ is infinitesimal and therefore $\cos\theta = 1$ and $\sin\theta = \theta = x/n$. Substituting these into the previous equation produces:

$$\cos x = \cos n\theta = \frac{(1+\frac{ix}{n})^n + (1-\frac{ix}{n})^n}{2}.$$

Now, Euler considered $e^\omega = 1+\omega$ for ω infinitesimally small. With this in mind, if a is some finite number and n is infinitely big,

$$e^a = (e^{a/n})^n = \left(1+\frac{a}{n}\right)^n.$$

Plugging in ix and $-ix$ in for a in that equation and substituting into the $\cos x$ formula,

$$\cos x = \frac{e^{ix}+e^{-ix}}{2}.$$

Now we have to work with the sine function. Using the same procedure as before,

$$\sin x = \sin n\theta = \frac{(1+\frac{ix}{n})^n - (1-\frac{ix}{n})^n}{2i} = \frac{e^{ix}+e^{-ix}}{2i}.$$

Adding these two results together leads to:

$$\cos x + i\sin x = \frac{e^{ix}+e^{-ix}}{2} + i\frac{e^{ix}+e^{-ix}}{2i} = \frac{2e^{ix}}{2} = e^{ix}.$$

□

Proof 2. Euler let $y = \sin x$ and therefore $x = \arcsin y = \int \frac{dy}{\sqrt{1-y^2}}$ Now letting $y = iz$ and $dy = idz$ Euler got

$$x = \int \frac{idz}{\sqrt{1-(iz)^2}} = i\int \frac{dz}{\sqrt{1+z^2}} = i\log(\sqrt{1+z^2}+z)$$

(which can be gotten from trigonometric substitution with $z = \tan\theta$). Euler then realized that $z = y/i = \sin x/i$ so $z^2 = \sin^2 x/i^2 = -\sin^2 x$. Therefore:

$$x = i\log\left(\sqrt{1-\sin^2 x}\right) + \frac{\sin x}{i} = i\log(\cos x - i\sin x).$$

Multiplying through by i:

$$ix = i^2\log(\cos x - i\sin x) = \log\left(\frac{1}{\cos x - i\sin x}\right) = \log(\cos x + i\sin x).$$

Exponentiating both sides:

$$e^{ix} = e^{\log(\cos x + i\sin x)} = \cos x + i\sin x.$$

□

Proof 3. This proof will be a direct substitution of ix into the power series expression for e^x The power series representation of e^x is

$$e^x = 1 + x + \frac{x^2}{1\cdot 2} + \frac{x^3}{1\cdot 2\cdot 3} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} + \cdots .$$

Substituting ix in for x and grouping terms together, Euler's formula pops right out.

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{(ix)^2}{1\cdot 2} + \frac{(ix)^3}{1\cdot 2\cdot 3} + \frac{(ix)^4}{1\cdot 2\cdot 3\cdot 4} + \frac{(ix)^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \cdots \\ &= 1 + ix - \frac{x^2}{1\cdot 2} - \frac{ix^3}{1\cdot 2\cdot 3} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} + \frac{ix^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \cdots \\ &= \left[1 - \frac{x^2}{1\cdot 2} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} - \cdots\right] + i\left[x - \frac{x^3}{1\cdot 2\cdot 3} + \frac{x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} \cdots\right] \\ &= \cos x + i \sin x \end{aligned}$$

[1] □

Euler is considered one of the greatest mathematicians that the world has ever seen. His insight and contributions are ubiquitous and can be found in just about every area of mathematics. Not only did he impact the mathematics of the era, he even developed new fields of study including analytic number theory and graph theory (not covered in this paper). On the other end of the spectrum, he added insight into the ideas of ancient civilizations, including the study of perfect and amicable numbers. Euler was able to make some of the biggest breakthroughs in mathematics, even though his sight started to deteriorate when he was around thirty years old. Euler did not just work in pure mathematics; he revolutionized the sciences as well with contributions to mechanics and other physical sciences.

As Laplace eloquently stated,"Euler is the master of us all".

References

[1] Dunham, William. *Euler: The Master of Us All.* Washington DC: The Mathematical Association of America, 1999.

[2] Dunham, William. *The Genius of Euler.* Washington DC: The Mathematical Association of America, 2007.

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