

MIDTERM 2 REVIEW SHEET

1. TRIGONOMETRIC SUBSTITUTION

- ◇ In these types of problems, you will see a term involving $a^2 - x^2$, $a^2 + x^2$, or $x^2 - a^2$, where $a > 0$ is some number. Typically, you will see the square root of one of these terms, but in some problems, you may the term raised to a different power (other than $\frac{1}{2}$).
- ◇ In any case, the table below tells you how to handle these types of integrals.

EXPRESSION	SUBSTITUTION	IDENTITY	RESULT
$\sqrt{a^2 - x^2}$	$x = a \sin \theta$	$1 - \sin^2 \theta = \cos^2 \theta$	$a \cos \theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta$	$1 + \tan^2 \theta = \sec^2 \theta$	$a \sec \theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta$	$\sec^2 \theta - 1 = \tan^2 \theta$	$a \tan \theta$

- ◇ Always remember to substitute for each term in the integral, including the dx !

2. PARTIAL FRACTIONS

- ◇ Apply this method when dealing with rational functions of the form $\frac{P(x)}{Q(x)}$, where $\deg P(x) < \deg Q(x)$. If $\deg P(x) \geq \deg Q(x)$, use polynomial long division so that you obtain a rational function in which the degree of the numerator is less than the degree of the denominator.
- ◇ You should factor $Q(x)$ as far as possible, into a product of linear factors (of the form $ax + b$) and a product of irreducible quadratic factors (of the form $ax^2 + bx + c$). Use the following four scenarios to determine what the partial fraction decomposition should look like.

- ▷ If $Q(x)$ factors into a product of *distinct* linear factors, i.e.,

$$Q(x) = (a_1x + b_1)(a_2x + b_2) \cdots (a_kx + b_k)$$

the decomposition is

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}$$

- ▷ If $Q(x)$ factors into a product of linear factors, but some are repeated, include the term $\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_r}{(ax+b)^r}$ for each factor of the form $(ax + b)^r$.
- ▷ If $Q(x)$ contains *distinct* quadratic factors, include a term of the form $\frac{Ax+B}{ax^2+bx+c}$ for each quadratic factor $ax^2 + bx + c$.

- ▷ If $Q(x)$ contains repeated quadratic factors, you should include a term of the form $\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_rx+B_r}{(ax^2+bx+c)^r}$ for each factor $(ax^2 + bx + c)^r$.
- ◇ To find the A_i 's in the first two scenarios above, it's often easier to clear denominators and then substitute roots of $Q(x)$ for x .
- ◇ The formula $\int \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$ is often used in cases where $Q(x)$ contains quadratic factors.

3. IMPROPER INTEGRALS

◇ Type I Integrals

- ▷ If $\int_a^t f(x) dx$ exists for all $t \geq a$, $\int_a^\infty f(x) dx = \lim_{t \rightarrow \infty} \int_a^t f(x) dx$ if this limit exists.
- ▷ If $\int_t^b f(x) dx$ exists for all $t \leq b$, $\int_{-\infty}^b f(x) dx = \lim_{t \rightarrow -\infty} \int_t^b f(x) dx$ if this limit exists.
- ▷ If $\int_{-\infty}^a f(x) dx$ and $\int_a^\infty f(x) dx$ both converge, we can define

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^a f(x) dx + \int_a^{\infty} f(x) dx$$

◇ Type II Integrals

- ▷ If $f(x)$ is continuous on $[a, b)$, but discontinuous at b , define $\int_a^b f(x) dx = \lim_{t \rightarrow b^-} \int_a^t f(x) dx$ if this limit exists.
- ▷ If $f(x)$ is continuous on $(a, b]$, but discontinuous at a , define $\int_a^b f(x) dx = \lim_{t \rightarrow a^+} \int_t^b f(x) dx$ if this limit exists.
- ▷ If $f(x)$ is discontinuous at c , where $a < c < b$ and $\int_a^c f(x) dx$ and $\int_c^b f(x) dx$ are both convergent, define $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$.
- ◇ Remember that $\int_1^\infty \frac{1}{x^p} dx$ converges if $p > 1$ and diverges if $p \leq 1$.
- ◇ *Comparison Theorem:* Suppose $f(x)$ and $g(x)$ are continuous and $f(x) \geq g(x) \geq 0$ for $x \geq a$.
 - ▷ If $\int_a^\infty f(x) dx$ converges, so does $\int_a^\infty g(x) dx$.
 - ▷ If $\int_a^\infty g(x) dx$ diverges, so does $\int_a^\infty f(x) dx$.

4. SEQUENCES

- ◇ $\lim_{n \rightarrow \infty} a_n = L$ means we can get the terms of the sequence as close to L as we like by taking n sufficiently large. If L exists as a finite number, the sequence *converges*, otherwise it *diverges*.
- ◇ $\lim_{n \rightarrow \infty} a_n = \infty$ means we can get the terms of the sequence as large as we like by taking n sufficiently large. In other words, for any M , there exists N such that $a_n \geq M$ for all $n \geq N$.
- ◇ If $\lim_{x \rightarrow \infty} f(x) = L$ and $f(n) = a_n$, then $\lim_{n \rightarrow \infty} a_n = L$.
- ◇ Suppose $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ are convergent sequences and c is a real number. The following are true:
 - ▷ $\lim_{n \rightarrow \infty} (a_n \pm b_n) = \lim_{n \rightarrow \infty} a_n \pm \lim_{n \rightarrow \infty} b_n$

- ▷ $\lim_{n \rightarrow \infty} ca_n = c \lim_{n \rightarrow \infty} a_n$
 - ▷ $\lim_{n \rightarrow \infty} c = c$
 - ▷ $\lim_{n \rightarrow \infty} a_n b_n = \lim_{n \rightarrow \infty} a_n \cdot \lim_{n \rightarrow \infty} b_n$
 - ▷ $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n}$ as long as $\lim_{n \rightarrow \infty} b_n \neq 0$
 - ▷ $\lim_{n \rightarrow \infty} a_n^p = (\lim_{n \rightarrow \infty} a_n)^p$ whenever $p > 0$ and $a_n > 0$.
- ◇ *Squeeze Theorem:* Suppose $a_n \leq b_n \leq c_n$ eventually and $\lim_{n \rightarrow \infty} a_n = L = \lim_{n \rightarrow \infty} c_n$. Then $\lim_{n \rightarrow \infty} b_n = L$ as well.
- ◇ If $\lim_{n \rightarrow \infty} |a_n| = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.
- ◇ Consider the sequence $(r^n)_{n \geq 1}$. This converges for $-1 < r \leq 1$ and diverges otherwise.

Also

$$\lim_{n \rightarrow \infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

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