

## VOLUMES OF SOLIDS OF REVOLUTION

**Definition 0.1.** A *solid of revolution* is obtained when a region enclosed by several curves is rotated about some vertical or horizontal line, say  $x = a$  or  $y = c$ . To find the volume, we find a relevant formula for a typical cross section and then integrate this result over the region.

(1) Cross sections *perpendicular* to axis of revolution.

(i) *Disks* occur when the axis of revolution, i.e., the line we are rotating about, is one of the boundary curves for the region. In this case, there is no gap, and the area of a typical cross section is given by the formula  $A = \pi r^2$  where  $r$  is the radius of the cross section.

Case 1: Rotating about a horizontal line,  $y = c$ . Here, recall that  $|f(x) - c|$  just means the distance between our curve and the line we are rotating about.

$$V = \int_a^b \pi |f(x) - c|^2 dx \quad \text{or in the special case } c = 0 \quad V = \int_a^b \pi f(x)^2 dx$$

Case 2: Rotating about a vertical line,  $x = a$ . Here,  $|g(y) - a|$  is again just the distance between our curve and the line we are rotating about.

$$V = \int_c^d \pi |g(y) - a|^2 dy \quad \text{or in the special case } a = 0 \quad V = \int_c^d \pi g(y)^2 dy$$

(ii) *Washers* occur when there is a gap between the area enclosed by the curves and the line we are rotating about. The area of a typical cross section is  $A = \pi(R^2 - r^2)$ , where  $R$  is the large (outer) radius of our washer and  $r$  is the small (inner) radius.

Case 1: Rotating about a horizontal line,  $y = c$ .

$$V = \int_a^b \pi \left| |f(x) - c|^2 - |g(x) - c|^2 \right| dx \quad \text{or in the case } c = 0 \quad V = \int_a^b \pi |f(x)^2 - g(x)^2| dx$$

Case 2: Rotating about a vertical line,  $x = a$ .

$$V = \int_c^d \pi \left| |g(y) - a|^2 - |f(y) - a|^2 \right| dy \quad \text{or in the case } a = 0 \quad V = \int_c^d \pi |g(y)^2 - f(y)^2| dy$$

(2) Cross sections *parallel* to axis of revolution. In this case, the cross sections are called *cylindrical shells*.

Case 1: Rotating about a vertical line  $x = c$ . Here,  $|x - c|$  is just the distance between our cross section at  $x$  to the line we are rotating about. Suppose the region is bounded above by  $y = f(x)$  and below by  $y = g(x)$ .

$$V = \int_a^b 2\pi |x - c| (f(x) - g(x)) dx \quad \text{or in the case } c = 0 \quad V = \int_a^b 2\pi x (f(x) - g(x)) dx$$

Case 2: Rotating about a horizontal line  $y = a$ . Here,  $|y - a|$  is just the distance between our cross section at  $y$  to the line we are rotating about. Suppose the region is bounded on the right by  $x = g(y)$  and on the left by  $x = f(y)$ .

$$V = \int_c^d 2\pi |y - a| (g(y) - f(y)) dy \quad \text{or in the case } a = 0 \quad V = \int_c^d 2\pi y (g(y) - f(y)) dy$$