1. POWER SERIES

◇ A **power series about a** is a series of the form $\sum_{n=0}^{\infty} c_n(x-a)^n$. In the special case $a = 0$, this becomes $\sum_{n=0}^{\infty} c_n x^n$. Here, the c_n's are called **coefficients**. Power series may not converge for all values of x.

◇ Three things can occur:

(1) The series converges only for $x = a$;
(2) The series converges for all x; or
(3) The series converges if $|x-a| < R$ and diverges if $|x-a| > R$. This number R is called the **radius of convergence**.

◇ In most cases, the Ratio Test is used to find the radius of convergence, but other tests can be applied in certain cases. To find the **interval of convergence**, once must check the endpoints separately, i.e., the points where $|x-a| = R$.

◇ We can differentiate and integrate power series term-by-term. The radius of convergence will stay the same, but the interval of convergence may be different, i.e., what happens at the endpoints can change after differentiating or integrating.

2. TAYLOR AND MACLAURIN SERIES

◇ The **Taylor series for f about** a is the series $\sum_{n=0}^{\infty} c_n(x-a)^n$ where $c_n = \dfrac{f^{(n)}(a)}{n!}$.

◇ The **Maclaurin series** is the Taylor series in the case $a = 0$.

◇ If a function f has a power series expansion at a, then it must be the sum of its Taylor series about a, i.e., we must have $f(x) = \sum_{n=0}^{\infty} \dfrac{f^{(n)}(a)}{n!}(x-a)^n$.

◇ The n^{th} **degree Taylor polynomial of f at** a is given by $T_n(x) = \sum_{n=0}^{n} \dfrac{f^{(i)}(a)}{i!}(x-a)^i$.

◇ Write $f(x) = T_n(x) + R_n(x)$. If it's the case that $\lim_{n\to\infty} R_n(x) = 0$ for $|x-a| < R$, then f is equal to the sum of its Taylor series about a on the interval $|x-a| < R$.

◇ We can use **Taylor's Inequality** to say something about $R_n(x)$: Suppose $|f^{(n+1)}(x)| \leq M$ on the interval $|x-a| < R$. Then $|R_n(x)| \leq \dfrac{M}{(n+1)!}|x-a|^{n+1}$.

◇ Here we collect some important Maclaurin series and their intervals of convergence:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad (-1,1)$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (-\infty,\infty)$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad (-\infty,\infty)$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad (-\infty,\infty)$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)} \qquad [-1,1]$$

3. ARC LENGTH

◇ If f' is continuous on $[a,b]$, then the length of the curve $y = f(x)$ for $a \leq x \leq b$ is given by $L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx = \int_a^b \sqrt{1+(\frac{dy}{dx})^2}\,dx$.

◇ There are similar formulas if we are given $x = g(y)$ instead, for $c \leq y \leq d$.

4. SURFACE AREA

◇ Suppose we have some smooth curve and we rotate it about one of the axes. If we rotate about the x-axis, the surface area of the resulting solid is $S = \int 2\pi y\,ds$ and if we rotate about the y-axis, it's $S = \int 2\pi x\,ds$ where in each case, ds is either of the following:

$$ds = \begin{cases} \sqrt{1+(\frac{dy}{dx})^2}dx \\ \sqrt{1+(\frac{dx}{dy})^2}dy \end{cases}$$

5. PARAMETRIC EQUATIONS

◇ In some cases, x and y are given as a function of another parameter t, and a curve can be described by a set of **parametric equations** $x = f(t)$ and $y = g(t)$.

◇ We can find slopes of tangent lines to parametric curves without eliminating the parameter by using the formula

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

as long as $\frac{dx}{dt} \neq 0$.

◇ Horizontal tangents occur when $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$. Vertical tangents occur when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$.

- We can find the area enclosed by a parametric curve between $\alpha \leq t \leq \beta$ by using the formula:

$$A = \int_{\alpha}^{\beta} g(t) f'(t)\, dt$$

.

- We can also find arc length and surface area using parametric equations. Here, we insist that the curve be traversed only once over the interval $\alpha \leq t \leq \beta$:

$$L = \int_{\alpha}^{\beta} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt$$

- If we rotate about the x-axis, the surface area of the resulting solid is $S = \displaystyle\int 2\pi y\, ds$ and if we rotate about the y-axis, it's $S = \displaystyle\int 2\pi x\, ds$ where in this case, $ds = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt$

6. POLAR COORDINATES

- **Polar coordinates** are of the form (r, θ) where r is the distance from the origin and θ is the angle the point makes with the positive x-axis.
- We can convert between polar and cartesian coordinates using

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ r^2 &= x^2 + y^2 \\ \frac{y}{x} &= \tan\theta \end{aligned}$$

- We can compute tangents to polar curves $r = f(\theta)$ using the above formula for parametric equations. Write $x = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$. Then

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f'(\theta)\sin\theta + r\cos\theta}{f'(\theta)\cos\theta - r\sin\theta}$$

- The area enclosed by a polar curve is $A = \int_a^b \frac{1}{2}[f(\theta)]^2\, d\theta$.
- The length of a curve with polar equation $r = f(\theta)$, $a \leq \theta \leq b$ is given by $L = \int_a^b \sqrt{r^2 + (\frac{dr}{d\theta})^2}\, d\theta$.