Research Statement: Micah B. Milinovich

My research interests are in the field of analytic number theory, especially the theory of the Riemann zeta-function and other L-functions. Some of the problems in which I am interested include estimating moments of L-functions, investigating the distribution of the zeros of L-functions, and studying the behavior of L-functions near the central point or on the critical line.

1. DISCRETE MOMENTS OF THE DERIVATIVE OF THE RIEMANN ZETA-FUNCTION

Let $\zeta(s)$ denote the Riemann zeta-function. An important problem in analytic number theory asks for estimates of the moments

$$I_k(T) = \frac{1}{T}\int_0^T \left|\zeta(\tfrac{1}{2}+it)\right|^{2k}\, dt$$

when $k > 0$. Due to a flurry of recent results and conjectures in the subject, a clear picture of the behavior of moments $I_k(T)$ is now emerging. Many of the results in my thesis focus on establishing similar results for the analogous discrete moments

$$J_k(T) = \frac{1}{N(T)}\sum_{0<\gamma\le T} \left|\zeta'(\rho)\right|^{2k}$$

where $k \in \mathbb{R}$ and the sum runs over the non-trivial (complex) zeros $\rho = \beta + i\gamma$ of $\zeta(s)$. Here the function $N(T) \sim \frac{T}{2\pi}\log T$ denotes the number of zeros of $\zeta(s)$ up to a height T counted with multiplicity. It is an open question to determine the behavior of $J_k(T)$ as k varies. Independently, Gonek [12] and Hejhal [13] have made the following conjecture.

Conjecture. (Gonek, Hejhal) *For $k \in \mathbb{R}$ and sufficiently large T we have $J_k(T) \asymp (\log T)^{k(k+2)}$.*

Until recently, evidence in favor of this conjecture existed for only a few small values of k. Assuming the Riemann Hypothesis, Gonek [10] has shown that $J_1(T) \sim \frac{1}{12}(\log T)^3$ and Ng [23] has proved that $J_2(T) \asymp (\log T)^8$. Also, Gonek [12] has shown that $J_{-1}(T) \gg (\log T)^{-1}$. In my thesis, I consider the moments $J_k(T)$ for larger values of k and am able to prove the following theorem which provides strong evidence for the conjecture of Gonek and Hejhal in the case when $k \in \mathbb{N}$.

Theorem 1.1. *Assume the Riemann Hypothesis. Then for sufficiently large T we have*

$$(\log T)^{k(k+2)} \ll_k \frac{1}{N(T)}\sum_{0<\gamma\le T} \left|\zeta'(\rho)\right|^{2k} \ll_{k,\varepsilon} (\log T)^{k(k+2)+\varepsilon}$$

where $k \in \mathbb{N}$ and $\varepsilon > 0$ is arbitrary.

The proof of the lower bound in Theorem 1.1, which is joint work with Nathan Ng [20], is an adaptation of the recent method introduced by Rudnick and Soundararajan [9, 10] to study the central moments of L-functions averaged over families. The proof of the upper bound [19] follows from a delicate study of mean-value estimates for Dirichlet polynomials that approximate $\log|\zeta(s)|$ in the right half of the critical strip (including the critical line $\Re s = \frac{1}{2}$). This builds upon the previous work of Selberg [26, 27, 28] and Soundararajan [30].

The methods used to prove the upper bound for moments $J_k(T)$ are very robust and can be applied to many other situations where averages of L-functions and their derivatives are of interest. For instance, the following example, which estimates averages of $|\zeta(s)|$ at its extremum on the critical line, will appear in my thesis.

Theorem 1.2. *Assume the Riemann Hypothesis. Let $k \in \mathbb{N}$, $\varepsilon > 0$ be arbitrary, and let γ and γ^+ denote consecutive ordinates of zeros of $\zeta(s)$. Then for sufficiently large T we have*

$$(\log T)^{k^2-\varepsilon} \ll \frac{1}{N(T)} \sum_{0<\gamma\le T} \max_{\gamma\le\tau_\gamma\le\gamma^+} \left|\zeta(\tfrac{1}{2}+i\tau_\gamma)\right|^{2k} \ll (\log T)^{k^2+\varepsilon}$$

where the implied constants depend on k and ε.

Previously, estimates for moments of the above form as strong as those in Theorem 1.2 were only known for the cases $k = 1$ and 2.

In future work, I hope to modify these methods to handle the case of upper bounds for the moments of derivatives of L-functions at the central point averaged over families. An example that I have begun to consider are the moments of quadratic twists of derivatives of an L-function associated to a fixed elliptic curve. Such moments have been studied previously by Murty & Murty [22], Iwaniec [14], and others.

2. Lower-order terms for $J_k(T)$

The recent work of Conrey, Farmer, Keating, Rubinstien, and Snaith [5] and of Conrey, Farmer, and Zirnbauer [6] provides a general method to conjecturally determine precise mean-value estimates for the products and ratios of shifted values of the L-functions. This information can then be used to conjecture the lower-order terms for many interesting examples of moments of (unshifted) L-functions. In particular, Conrey and Snaith [7] have made some precise conjectures for the lower-order terms of the moments $J_k(T)$ when $k = 1$ or 2. In [18], I have verified their conjecture when $k = 1$ by estimating the integral

$$\mathcal{J} = \frac{1}{i}\int_{1+i}^{1+iT} \frac{\zeta(s+\alpha_1)\zeta(s+\alpha_2)}{\zeta(s+\alpha_3)} \, \frac{\zeta(1-s+\beta_1)\zeta(1-s+\beta_2)}{\zeta(1-s+\beta_3)} \, ds$$

both unconditionally and on the Riemann Hypothesis (RH). For suitably restricted shifts α_i, β_j $(i, j = 1, 2, 3)$, I have shown that

$$\mathcal{J} = \frac{1}{2\pi i}\int_{\mathscr{C}} \frac{\zeta(w+\alpha_1)\zeta(w+\alpha_2)}{\zeta(w+\alpha_3)} \, \frac{\zeta(w-\beta_1)\zeta(w-\beta_2)}{\zeta(w-\beta_3)} \Big(\int_0^T \big(\tfrac{t}{2\pi}\big)^{w-\beta_1-\beta_2-\beta_3-1} dt\Big) \, dw + E(T)$$

where $\mathscr{C}$ is any positively oriented closed contour that containing the poles of the integrand and the error term $E(T)$ is $o(T)$ unconditionally and $O(T^{1/2+\varepsilon})$ on RH. From this I am able to deduce the following theorem.

Theorem 2.1. *Assume the Riemann Hypothesis. Let $\mu \ge 0$ be an integer and let $\varepsilon > 0$ be arbitrary. Then there exists a polynomial $\mathcal{P}_\mu(\cdot)$ of degree $2\mu + 2$ and positive leading coefficient such that*

$$\sum_{0<\Im\rho\le T} \left|\zeta^{(\mu)}(\rho)\right|^2 = \frac{T}{2\pi}\mathcal{P}_\mu\big(\log\frac{T}{2\pi}\big) + O\big(T^{1/2+\varepsilon}\big).$$

The result of Theorem 2.1, in the case $\mu = 1$, confirms the conjecture of Conrey and Snaith mentioned above since my proof allows one to compute the coefficients of $\mathcal{P}_\mu(\cdot)$ explicitly.

3. Work in progress

Below is a summary of four projects that I have already begun to work on. I hope to include the problems described in 3.1 and 3.2 in my thesis.

3.1. **Value distributiton for $|\zeta(\frac{1}{2}+it)|$.** The functional equation for the Riemann zeta-function states that $\zeta(s)=\chi(s)\zeta(1-s)$ where $\chi(s)=2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)$. It follows from this that the function $Z(t)=\chi(\frac{1}{2}+it)^{-1/2}\zeta(\frac{1}{2}+it)$ is a real-valued, continuously differentiable function satisfying $|Z(t)|=|\zeta(\frac{1}{2}+it)|$. If the Riemann Hypothesis is true, the zeros of $Z(t)$ and $Z'(t)$ interlace and if the zeros of $\zeta(s)$ are simple then the zeros of $Z(t)$ and $Z'(t)$ are distinct. I would like to prove a Selberg-type value distribution theorem (see Section 2 of [28]) for $\log|\zeta(\frac{1}{2}+it)|=\log|Z(t)|$ at the zeros of $Z'(t)$. In order to do so we will likely have to make the following well-spacing conjecture between the zeros of γ of $Z(t)$ and the zeros γ_1 of $Z'(t)$.

Conjecture 3.1. *For T sufficiently large, there exists a ϑ with $0<\vartheta<1$ such that*

$$\frac{\#\{0<\gamma\le T : |\gamma-\gamma_1|\le 2\pi\delta/\log T\}}{N(T)} \ll \delta^{\vartheta}$$

uniformly as $\delta\to 0^+$.

Here, $N(T)$ is the same zero-counting function as in Section 1. This conjecture is similar to an assumption made by Bombieri and Hejhal [1] in their work on the zeros of linear combinations of L-functions. Let $\chi_{[a,b]}(\cdot)$ denote the characteristic function of the interval $[a,b]$, then it appears likely we can use Selberg's ideas to prove the following result.

Assume the Riemann Hypothesis, that the zeros of $\zeta(s)$ are simple, and Conjecture 3.1. Then for sufficiently large T we have

$$\frac{1}{N(T)}\sum_{0<\gamma_1\le T}\chi_{[\alpha,\beta]}\Big(\frac{\log|\zeta(\frac{1}{2}+i\gamma_1)|}{\sqrt{\pi\log\log T}}\Big)=\int_\alpha^\beta e^{-\pi x^2}\,dx+O\Big(\frac{(\log\log\log T)^A}{\sqrt{\log\log T}}\Big)$$

for some absolute constant $A>0$.

This result could be used to study the function $N_\alpha(T)=\#\{0<\tau_\alpha\le T : |\zeta(\frac{1}{2}+i\tau_\alpha)|=\alpha\}$. If $\varepsilon>0$ and $0\le\alpha\le(\log\log T)^{1/2-\varepsilon}$, then under the same assumptions as above it appears likely we can show that

$$N_\alpha(T)=\frac{T}{2\pi}\log T+O\Big(T\log T\frac{(\log\log\log T)^A}{\sqrt{\log\log T}}\Big)$$

for some absolute constant $A>0$.

3.2. **Spacing between the zeros of $Z(t)$ and $Z'(t)$.** Graphical and numerical evidence suggests that zeros of $Z'(t)$ should occur near the mid-point of two consecutive zeros of $Z(t)$. Since the zeros of $Z(t)$ are expected to repel one another, this would suggest that there is repulsion between the zeros of $Z(t)$ and $Z'(t)$, as well. I am currently exploring this phenomenon using techniques similar to Montgomery's study [21] of the pair correlation of the imaginary parts of zeros of the Riemann zeta-function. The goal is to determine the behavior of sums of the form

$$\sum_{0<\gamma,\gamma_1\le T}F\big((\gamma-\gamma_1)\log\tfrac{T}{2\pi}\big)$$

for functions $F(\cdot)$ whose Fourier transforms are compactly supported. This should provide evidence for the well-spacing Conjecture 3.1.

3.3. **Pair correlation of the zeros of $\Re e\ \xi(\sigma+it)$.** This project intends to study the zeros of the function $\Re e\ \xi(\sigma+it)$ where $\xi(s)=\frac{1}{2}s(1-s)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ is the usual Riemann ξ-function. If the Riemann Hypothesis is true, then when $\sigma=1/2$ the zeros of this function correspond to the zeros of $\zeta(s)$ and Montgomery [21] has conjectured the pair correlation of these zeros. When $\sigma>1/2$ and is fixed, Lagarias [16] has shown that the zeros of $\Re e\ \xi(\sigma+it)$ have a trivial limiting distribution; that is, when appropriately normalized, the distribution corresponds to that of the distribution of a set of points equally spaced by one. This project intends to study how the pair correlation function changes from trivial to GUE (Montgomery's prediction) as $\sigma\to 1/2$.

3.4. **Discrete mean-value estimates of long Dirichlet polynomials.** This project is joint J. Bian, H. Hahn, S. Gonek, and N. Ng. We will attempt to generalize Goldston and Gonek's results [11] concerning continuous mean-values of long Dirichlet polynomials to the case of mean-values of long Dirichlet polynomials averaged over the zeros of $\zeta(s)$ or, possibly, any L-function in the Selberg class (see [28] for the definition). By long Dirichlet polynomial we simply mean a Dirichlet polynomial that is longer than the range of summation. The evaluation of these mean-values theorems involves a delicate analysis of the off-diagonal terms. In this case, it appears that these discrete mean-value theorems are intimately connected to correlation sums of the form $\sum_{n<x}a(n)a(qn+h)$ where $q=p^k$ for some prime p. When $a(n)$ is a divisor-type function, then these sums are reminiscent to the "quadratic divisor problem" studied by Duke, Friedlander, and Iwaniec [9]. The goal of this project is to make the connection between the correlation sums and the mean-values as precise as possible and also to illustrate the usefulness of such mean-value theorems to the study of gaps between consecutive zeros of $\zeta(s)$.

4. Future work

In addition to the problems described above, I have begun to think about a few projects relating to the distribution of the low-lying of L-functions and their derivatives. Two of these projects are related to the work of Soundararajan [29] and Conrey & Soundararajan [8] on the mollified moments of quadratic Dirichlet L-functions. An additional project intends to initiate a study of the distribution of the low-lying zeros of the derivatives of completed L-functions using the techniques described by Katz & Sarnak in [15].

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