On the state of Wan's Conjecture

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Laurent Polynomials

Let $q = p^a$ where p is a prime and a is a positive integer. Let $\mathbb{F}_q$ denote the field of q elements.
For a Laurent polynomial $f \in \mathbb{F}_q[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ we may represent f as:

$$f = \sum_{j=1}^{J} a_j x^{V_j}, a_j \neq 0,$$

where each exponent $V_j = (v_{1j}, \ldots, v_{nj})$ is a lattice point in $\mathbb{Z}^n$ and the power x^{V_j} is the product $x_1^{v_{1j}} \cdot \ldots \cdot x_n^{v_{nj}}$.

Example

$$\begin{array}{rcccccc} f(x_1, x_2) & = & \frac{2}{x_1} & + & 10x_1x_2^2 & + & 82 \\ \text{lattice points} & = & \{(-1, 0) & , & (1, 2) & , & (0, 0)\} \end{array}$$

$\mathbb{F}_p(\Delta)$

Let $\Delta(f)$ denote Newton polyhedron of f, that is, the convex closure of the origin and $\{V_1, \ldots, V_J\}$, the integral exponents of f.

Definition

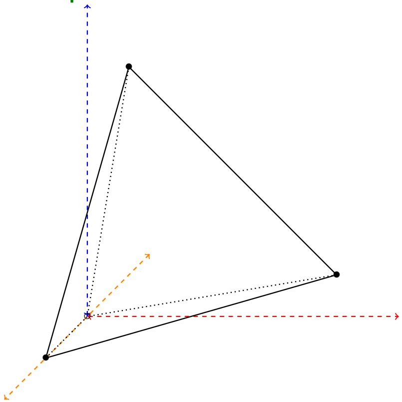
Given a convex integral polytope Δ which contains the origin, let $\mathbb{F}_q(\Delta)$ be the space of functions generated by the monomials in Δ with coefficients in the algebraic closure of $\mathbb{F}_q$, a field of q elements.

In other words,

$$\mathbb{F}_q(\Delta) = \{f \in \mathbb{F}_q[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \mid \Delta(f) \subseteq \Delta\}.$$

The polytope Δ

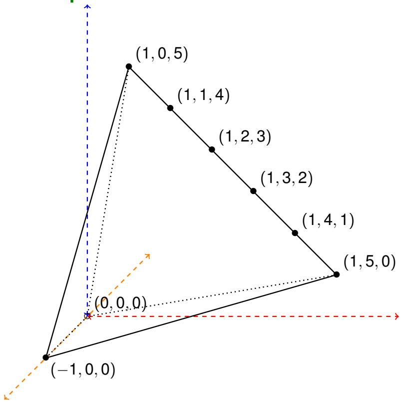
Example



Let Δ be the polytope generated by $f(x, y, z) = 1/z + x^5 z + y^5 z$.

The polytope Δ

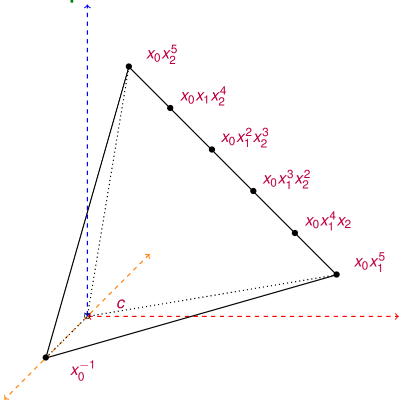
Example



It is also the convex closure of the lattice points (including interior points).

The polytope Δ

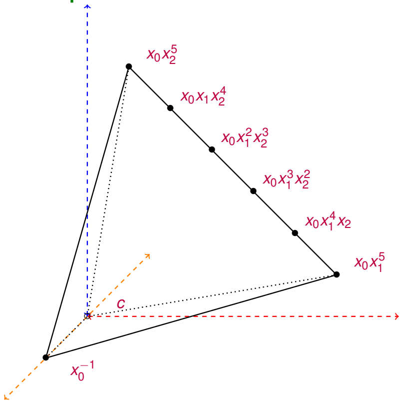
Example



We can correspond each lattice point to a monomial in *n* variables (including interior points).

The polytope Δ

Example



$\mathbb{F}_p(\Delta)$ is space of functions the generated by these monomials (including interior points).

$M_q(\Delta)$

Definition

The Laurent polynomial f is called non-degenerate if for each closed face δ of $\Delta(f)$ of arbitrary dimension which does not contain the origin, the n partial derivatives

$$\{\frac{\partial f_\delta}{\partial x_1}, \ldots, \frac{\partial f_\delta}{\partial x_n}\}$$

have no common zeros with $x_1 \cdots x_n \neq 0$ over the algebraic closure of $\mathbb{F}_q$.

Definition

Let $M_q(\Delta)$ be the functions in $\mathbb{F}_q(\Delta)$ that are non-degenerate.

Definition of the L-function

Let $f \in \mathbb{F}_q[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Let ζ_p be a p-th root of unity and $q = p^a$. For each positive integer k, consider the exponential sum:

$$S_k^*(f) = \sum_{(x_1,\ldots,x_n) \in \mathbb{F}_{q^k}^*} \zeta_p^{Tr_k f(x_1,\ldots,x_n)}.$$

The behavior of $S_k^*(f)$ as k increases is difficult to understand.

L-function

To better understand $S_k^*(f)$ we define the *L*-function as follows:

$$\begin{array}{ccccc} \mathbb{F}_q, & \mathbb{F}_{q^2}, & \dots & \mathbb{F}_{q^k}, & \dots \\ S_1^*(f), & S_2^*(f), & \dots & S_k^*(f), & \dots \\ S_1^*(f)T + & S_2^*(f)\frac{T^2}{2} + & \dots + & S_k^*(f)\frac{T^k}{k} + & \dots \end{array}$$

$$L^*(f,T) = \exp\left(\sum_{k=1}^{\infty} S_k^*(f)\frac{T^k}{k}\right).$$

By a theorem of Dwork-Bombieri-Grothendieck $L(f,T)$ is a rational function.

$NP(f)$

Adolphson and Sperber showed that if f is non-degenerate

$$L^*(f,T)^{(-1)^{n-1}} = \sum_{i=0}^{\infty} A_i(f)T^i, \quad A_i(f) \in \mathbb{Z}[\zeta_p]$$

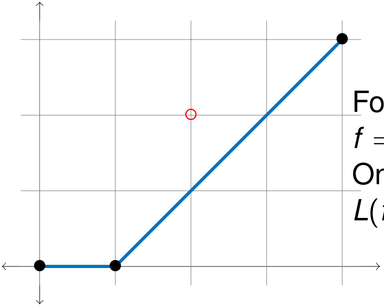
is a polynomial of degree $n!Vol(\Delta)$.

Definition

Define the Newton polygon of f, denoted $NP(f)$ to be the lower convex closure in $\mathbb{R}^2$ of the points

$$(k, \mathrm{ord}_q A_k(f)), k = 0, 1, \ldots, n!Vol(\Delta).$$

Example



For $p = q = 3$ and $f = \frac{1}{x_1} + x_1 x_2^2 + x_1 x_3^2$.
One can computed directly:
$L(f, T)^{-1} =$

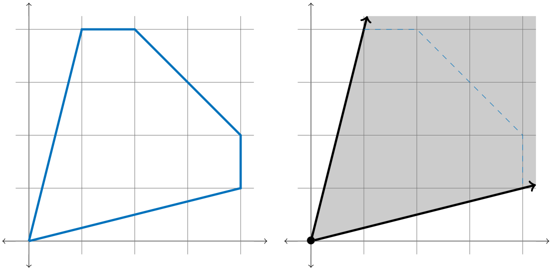
$$-27T^4 + 0T^3 + 18T^2 + 8T + 1$$

$(4,3)$	$(3,\infty)$	$(2,2)$	$(1,0)$	$(0,0)$

The Hodge Polygon

There exists a combinatorial lower bound to the Newton polygon called the Hodge polygon $HP(\Delta)$. This is constructed using the cone generated by Δ consisting of all rays passing through nonzero points of Δ emanating from the origin.

Example



Main Question

Definition

When $NP(f) = HP(\Delta)$ we say f is **ordinary**.

Generic Newton Polygon

Let $GNP(\Delta, p) = \inf_{f \in M_p(\Delta)} NP(f)$.

Adophson and Sperber showed that $GNP(\Delta, p) \geq HP(\Delta)$ for every p.

Generic Ordinarity

Main Question
When is $GNP(\Delta, p) = HP(\Delta)$?

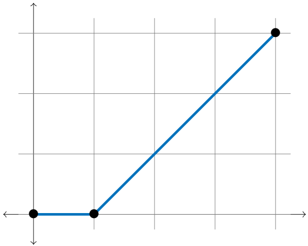
If $GNP(\Delta, p) = HP(\Delta)$ we say Δ is **generically ordinary** at p.

Adolphson and Sperber conjectured that if $p \equiv 1 \pmod{D(\Delta)}$ the $M_p(\Delta)$ is generically ordinary.
Wan showed that this is not quite true, but if we replace $D(\Delta)$ with an effectively computable $D^*(\Delta)$ this is true.

Wan's Conjecture

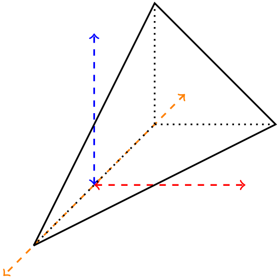
$$\lim_{p \to \infty} GNP(\Delta, p) = HP(\Delta)$$

Example of Ordinarity



Recall for $p = q = 3$ and $f = \frac{1}{x_1} + x_1 x_2^2 + x_1 x_3^2$, the Newton polygon of $L(f, T)^{(-1)^{(n-1)}} = -27T^4 + 18T^2 + 8T + 1$.

Example



- The Newton polygon $\Delta(f)$ the polytope spanned by the origin, $(-1,0,0)$, $(1,2,0)$ and $(1,0,2)$.
- $HP(\Delta(f))$ is the lower convex hull of the points $(0,0),(1,0)$ and $(4,3)$ which is identical to $NP(f)$.
- From this we see that the Newton Polygon is equal to the Hodge polygon. Hence f is ordinary.

- In 2002 Zhu showed that Wan's Conjecture holds for the one variable case.
- This was done by considering a specific family $x^d + ax$.
- Through direct computation she found the Generic Newton Polygon to be the lower convex hull of the points

$$(n, \frac{n(n+1)}{2d} + \epsilon_n)$$

Where

$$\lim_{p \to \infty} \epsilon_n = 0$$

- The Hodge polygon can be shown to be the lower convex hull of the points:

$$(n, \frac{n(n+1)}{2d})$$

- In 2004 Regis Blache showed that Wan's Conjecture holds for families of the form:

$$a_{d_1 1}x_1^{d_1} + a_{d_1 - 1 1}x_1^{d_1 - 1} + \ldots + a_{01}$$

$$+a_{d_2 2}x_2^{d_2} + a_{d_2 - 1 2}x_2^{d_2 - 1} + \ldots + a_{02}$$

$$\vdots$$

$$+a_{d_n n}x_n^{d_n} + a_{d_n - 1 n}x_n^{d_n - 1} + \ldots + a_{0n}$$

- These are families of polynomials with no cross terms like $x_1 x_2$.
- This was accomplished primarily by 'factoring' the Newton Polygon by variable. That is, he reduced this special multivariable case into the single variable case.
- He also addressed 'rectangular' families such as those generated by the polytope $(0,0), (d_1, 0), (0, d_2), (d_1, d_2)$.

- Last year Liu tackled these two specific families:

$$a_{(3,0)}x_1^3 + a_{(0,3)}x_2^3 + a_{(1,2)}x_1x_2^2 + a_{(2,1)}x_2x_2^1 + a_{(1,1)}x_1x_2^1$$

$$+a_{(2,0)}x_1^2 + +a_{(0,2)}x_2^2 + a_{(1,0)}x_1 + a_{(0,1)}x_2 + a_{(0,0)}$$

and

$$a_{(3,0)}x_1^3 + a_{(1,1)}x_1x_2^1 + a_{(2,0)}x_1^2 + +a_{(0,2)}x_2^2 + a_{(1,0)}x_1$$

$$+a_{(0,1)}x_2 + a_{(0,0)}$$

- This is an isosceles right triangle with leg length 3, and a leg length 2 isosceles right triangle with an additional point at $(3, 0)$.
- This was done in an entirely brute force method, computing the Newton Polygon speficially for these two families and showing that they tend toward the Hodge Polygon as p tends to infinity:

For the family:

$$a_{(3,0)}x_1^3 + a_{(0,3)}x_2^3 + a_{(1,2)}x_1x_2^2 + a_{(2,1)}x_2x_2^1 + a_{(1,1)}x_1x_2^1$$

$$+a_{(2,0)}x_1^2 + +a_{(0,2)}x_2^2 + a_{(1,0)}x_1 + a_{(0,1)}x_2 + a_{(0,0)}$$

For $p > 9$ and $p \equiv 2 \pmod 3$ the generic Newton Polygon is found to be:

$$(0,0),(1,0),(3,\frac{2p+2}{3(p-1)},(5,2),(6,\frac{8p-7}{3(p-1)}),$$

$$(8,\frac{14p-13}{3(p-1)}),(9,6)$$

For the family:

$$a_{(3,0)}x_1^3 + a_{(1,1)}x_1x_2^1 + a_{(2,0)}x_1^2 + +a_{(0,2)}x_2^2 + a_{(1,0)}x_1$$

$$+a_{(0,1)}x_2 + a_{(0,0)}$$

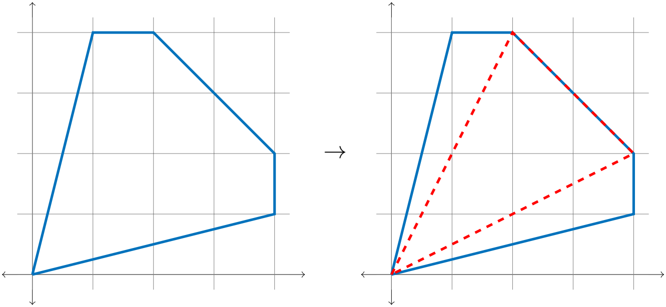
For $p > 18$ and $p \equiv 2 \pmod 3$ the generic Newton Polygon is found to be:

$$(0,0),(1,0),(2,\frac{p+1}{3(p-1)}),(3,\frac{5p-1}{6(p-1)}),(4,\frac{3p-1}{2(p-1)}),$$

$$(5,\frac{7p-2}{3(p-1)}),(6,\frac{7}{2})$$

A Decomposition of the Polytope

Wan and Le showed that certain decompositions will also decompose ordinarity.



Facial Decomposition

Let $\{\sigma_1, \ldots, \sigma_h\}$ be the set of faces of Δ that do not contain the origin.

Theorem (Facial Decomposition Theorem)

Let f be non-degenerate and let $\Delta(f)$ be n-dimensional. Then f is ordinary if and only if each f_{σ_i} is ordinary. Equivalently, f is non-ordinary if and only if if some f_{σ_i} is non-ordinary.

Using the facial decomposition theorem we may assume that $\Delta(f)$ is generated by a single codimension 1 face not containing the origin.
This allows us to concentrate on methods to decompose the individual faces of Δ.

Coherent Decomposition

Let δ be a face of Δ not containing the origin.

Definition

A **coherent** decomposition of δ is a decomposition T into polytopes $\delta_1, \ldots, \delta_h$ such that there is a piecewise linear function $\phi : \delta \mapsto \mathbb{R}$ such that

1. ϕ is concave i.e. $\phi(tx + (1-t)x') \geq t\phi(x) + (1-t)\phi(x')$, for all $x, x' \in \delta, 0 \leq t \leq 1$.
2. The domains of linearity of ϕ are precisely the n-dimensional simplices δ_i for $1 \leq i \leq m$.

Coherent decompositions are sometimes called concave decompositions.

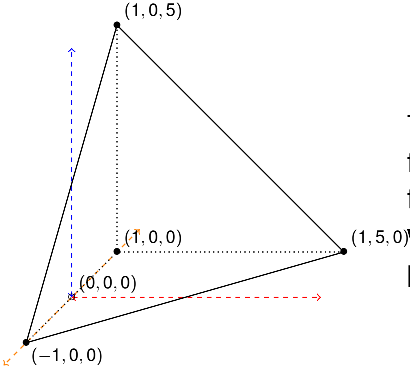
Coherent Decomposition Theorem

Let Δ be a polytope containing a unique face δ away from the origin. Let $\delta = \cup\delta_i$ be a complete coherent decomposition of δ. Let Δ_i denoted the convex closure of δ_i and the origin. Then $\Delta = \cup\Delta_i$. We call this a coherent decomposition of Δ.

Theorem (Coherent Decomposition (L-))

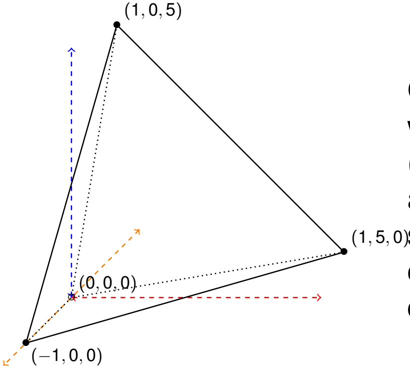
Suppose each lattice point of δ is a vertex of δ_i for some i. If each f_{Δ_i} is generically non-degenerate and ordinary for some prime p, then f is also generically non-degenerate and ordinary for the same prime p.

Example



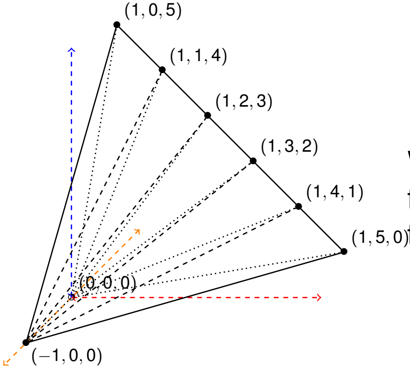
There are two faces away from the origin. Using the facial decomposition theorem we can divide this into two polytopes.

Example



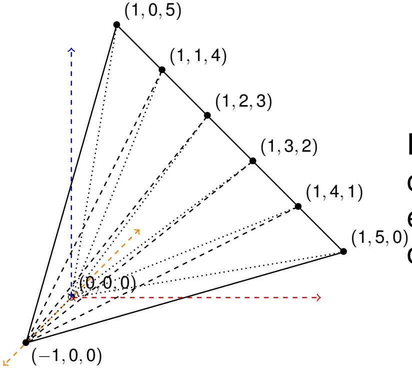
Consider the polytope Δ' with vertices $(0,0,0),(-1,0,0),(1,5,0)$ and $(1,0,5)$. Wan's work has shown that the back face is ordinary for any prime so we can ignore it.

Example



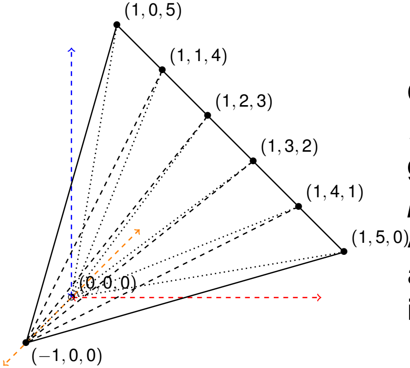
We can decompose the front face, which will decompose the entire polytope

Example



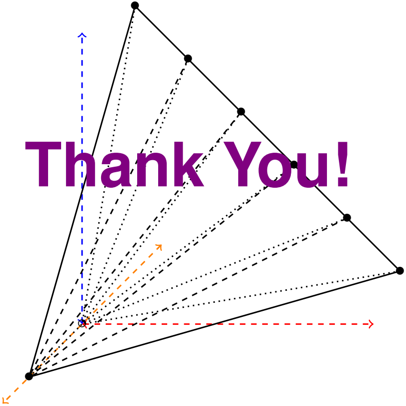
For any $f \in M_p(\Delta')$ if f is ordinary when restricted to each of these pieces, it is ordinary on all of f.

Example



One can show that $D(\Delta') = 5$ and Δ' is generically ordinary when $p \equiv 1 \pmod 5$, that is, Adolphson and Sperber's and Wan's conjecture holds in this case.

The End



Thank You!