The dynamical Mordell–Lang conjecture for Linear Maps

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Outline

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The Usual Setup

- S is a set (such as $\mathbb{Z}$ or $\mathbb{C}^g$).
- $f : S \to S$ is a self-map of S.
- $q \in S$
- The **orbit set of** q **under** f is

$$\{q, f(q), f(f(q)), f(f(f(q)),), \ldots\}$$

  or more concisely, $\mathrm{Orb}_f(q) := \{f^n(q) \mid n \in \mathbb{N}\}$ where $f^n := f(f^{n-1})$ is the n-fold composition of f with itself.
- Study $\mathrm{Orb}_f(q)$ and make interesting conclusions for various S, f, and q.

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Intersections of Orbit Sets with Curves

- Let $S = \mathbb{C}^2, f(x,y)$ be defined by polynomials in each coordinate, $\mathbf{q} \in \mathbb{C}^2$, and consider $\mathrm{Orb}_f(\mathbf{q})$.
- What can be said about $\mathrm{Orb}_f(\mathbf{q}) \cap C$ for some curve C?
- Example: Let $S = \mathbb{C}^2, f(x,y) = (ax+by, cx+dy)$ be a self-map of S, $\mathbf{q} = (q_1, q_2)$, and C is a curve of degree d.
  If the orbit set has finite intersection with C then there at most $(2N)^{35N^3}$ points in the intersection where $N = (d+1)^2$.
  When $d = 1$, this provides a uniform upper bound of 8^{2240}.
  Better bounds exist in special cases.

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The Eigenvalues of a linear map f

- $f : \mathbb{C}^2 \to \mathbb{C}^2$ is a linear map defined as $f(x, y) := (ax + by, cx + dy)$ for some $a, b, c, d \in \mathbb{C}$.
- $M := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Mat_{2\times 2}(\mathbb{R})$ is the associated matrix of f since $f^n(x, y) = M^n \cdot \begin{bmatrix} x \\ y \end{bmatrix}$ where $f^n := f^{n-1} \circ f$.
- The eigenvalues of f are the eigenvalues of M.
- For a linear map $f : \mathbb{C}^g \to \mathbb{C}^g$, we may also speak of the associated eigenvalues which arise from the $g \times g$ matrix of coefficients.

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The Dynamical–Mordell Lang Conjecture

$f = (f_1, \ldots f_g)$, $S = \mathbb{C}^g$, and $\mathbf{q} = (q_1, \ldots, q_g)$.

Conjecture (Ghioca, Tucker, Zieve 2007)

Let $f_1, \ldots, f_g$ be polynomials in $\mathbb{C}[x_1, \ldots, x_g]$ and let V be a subvariety of $\mathbb{C}^g$ which contains no positive dimensional subvariety that is periodic under the action of $(f_1, \ldots, f_g)$ on $\mathbb{C}^g$. Then V has finite intersection with each orbit of $(f_1, \ldots, f_g)$ on $\mathbb{C}^g$.

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Let $f_1, \ldots, f_g$ be linear polynomials in $\mathbb{C}[x_1, \ldots, x_g]$ of the form $f_i(x) = a_{i,1}x_1 + \cdots + a_{i,g}x_g$ and let V be a subvariety of $\mathbb{C}^g$ which contains no positive dimensional subvariety that is periodic under the action of $(f_1, \ldots, f_g)$ on $\mathbb{C}^g$. Then the number of points in the intersection of V and an orbit of $(f_1, \ldots, f_g)$ on $\mathbb{C}^g$ is at most $(2N)^{35N^3}$ where $N = (d+1)^g$.

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Strategy

- Parameterize the coordinates of the points, P_n, in the orbit set, $\mathrm{Orb}_f(\mathbf{q})$, as $P_n = (h_1(n), \ldots, h_g(n))$ for suitable functions $h_i(n)$.
- If $P_n \in V = Z\left(\sum_{\substack{i_1+\cdots+i_g \le d \\ i_1,\ldots,i_g \ge 0}} a_{i_1,\ldots,i_g} x_1^{i_1} \cdots x_g^{i_g} \right)$ then
$$\sum_{\substack{i_1+\cdots+i_g \le d \\ i_1,\ldots,i_g \ge 0}} a_{i_1,\ldots,i_g} (h_1(n))^{i_1} \cdots (h_g(n))^{i_g} = 0.$$
- The last summation will be a polynomial-exponential sum in the variable n whose order is N which depends on g and d.
- Apply a result due to Schlickewei for the maximum number of zeroes within a polynomial-exponential sum of order N.

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Polynomial-Exponential Sums

- A polynomial-exponential sum is a summation with the form

$$E(x) := \sum_{i=1}^{m} (P_i(x) b_i^x)$$

where $P_i(x) \in k[x]$ and $b_i \in k$ for some field k.

- The order of a poly-exp sum is $m + \sum_{i=1}^{m} \deg(P_i)$.
- These poly-exp sums show up in linear recurrences and the orbit set problem.

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Recurrence Sequences

- A linear recurrence sequence of order N over a field k is a sequence, $\{a_n\}_{n\in\mathbb{N}}$, of the form

$$a_{n+N} := \alpha_1 a_{n+N-1} + \alpha_2 a_{n+N-2} + \cdots + \alpha_N a_n$$

for $n \geq 0$ with initial values $(a_0, a_1, \ldots, a_{N-1}) := (\beta_0, \beta_1, \ldots, \beta_{N-1})$ for some $\alpha_i, \beta_i \in k$ and $\alpha_N \neq 0$. (or just N-ary recurrence sequence over k for short).

- Characteristic polynomial $x^N - \alpha_1 x^{N-1} - \alpha_2 x^{N-2} - \cdots - \alpha_N$ with roots $r_1, r_2, \ldots, r_m$ with r_i having multiplicity m_i so that

$$a_n = \sum_{i=1}^{m} \left(\sum_{j=1}^{m_i} c_{i,j} n^{j-1} \right) r_i^n.$$

- A recurrence sequence is non-degenerate if it takes on the value 0 finitely many times.

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Example of a Recurrence Sequence of Order 2

- $a_{n+2} := a_{n+1} + a_n$ with $(a_0, a_1) := (0, 1)$.
- $\{a_n\}_{n \in \mathbb{N}} = \{0, 1, 1, 2, 3, 5, 8, \ldots\}$.
- Characteristic polynomial is $x^2 - x - 1$ whose roots are $\frac{1 \pm \sqrt{5}}{2}$.
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$$\begin{aligned} a_n &= c_1 \left(\frac{1+\sqrt{5}}{2} \right)^n + c_2 \left(\frac{1-\sqrt{5}}{2} \right)^n \\ a_0 &= 0 \\ a_1 &= 1 \end{aligned}$$
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Connections to Linear Recurrences

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- If V has degree d then the linear recurrence will have order at most $(d+1)^g$.
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Skolem-Mahler-Lech Theorem



Theorem (Skolem-Mahler-Lech 1933-1935-1953)

If $\{a_n\}_{n\in\mathbb{N}}$ is a recurrence sequence of complex numbers, then the set of all integers n such that $a_n = 0$ is the union of a finite number of arithmetic sequences.

Arithmetic Sequences

- An arithmetic sequence of natural numbers is a sequence, $\{a_n\}_{n\in\mathbb{N}}$, of the form
$$a_n := s + nt$$
for some fixed $s, t \in \mathbb{N}$ and with $n \in \mathbb{N}$.
- If $t = 0$, then the arithmetic sequence is a singleton.
- If $t \neq 0$, then the arithmetic sequence is said to be a full arithmetic sequence (contains infinitely many numbers).
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Ternary Recurrence Theorems

Theorem (Beukers 1991)

If $\{a_n\}_{n \in \mathbb{N}}$ is a non-degenerate ternary recurrence sequence of rational numbers, then there are at most 6 integers n such that $a_n = 0$.

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If $\{a_n\}_{n \in \mathbb{N}}$ is a non-degenerate ternary recurrence sequence of complex numbers, then there are at most 61 integers n such that $a_n = 0$.

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N-ary Recurrence Theorems

Theorem (Schlickewei 2000)

If $\{a_n\}_{n\in\mathbb{N}}$ is a non-degenerate N-ary recurrence sequence of complex numbers, then there are at most $(2N)^{35N^3}$ integers n such that $a_n = 0$.

Theorem (D. 2010)

For $N > 1$, if $\{a_n\}_{n\in\mathbb{N}}$ is a non-degenerate N-ary recurrence sequence of real numbers whose characteristic roots are all real, then there are at most $2N - 3$ integers n such that $a_n = 0$.

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Orbit Sets and Varieties

Analyze $\{n \in \mathbb{N} \mid f^n(\mathbf{q}) \in W\}$ where $f : V \to V$, $\mathbf{q} \in V$, and W is a subvariety of $V := \bigcap_{i=1}^{m} Z\left(P_i(\vec{x})\right)$.

Theorem (Bell 2006)

Let V be an affine variety over a field k of characteristic 0. Let $\mathbf{q}$ be a point in V and f an automorphism of V. If W is a subvariety of V then the set $\{n \in \mathbb{N} \mid f^n(\mathbf{q}) \in W\}$ is a finite union of arithmetic sequences.

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Theorem (Bell, Ghioca, Tucker 2009)

Let $f : V \to V$ be an ètale endomorphism of any quasiprojective variety defined over $\mathbb{C}$. Then for any subvariety W of V, and for any point $\mathbf{q} \in V$ the set $\{n \in \mathbb{N} \mid f^n(\mathbf{q}) \in W\}$ is a finite union of arithmetic sequences.

Theorem (D. 2010)

If the eigenvalues of a linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$ are real, $\mathbf{q} \in \mathbb{R}^2$, $C \subset \mathbb{R}^2$ is a curve of degree d, and $|\mathrm{Orb}_f(q) \bigcap C|$ is finite, then $|\mathrm{Orb}_f(\mathbf{q}) \bigcap C| \leq d^2 + 3d - 1$.

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Dynamical–Mordell Lang for Linear Maps

Conjecture (D. 2012)

Let $f_1, \ldots, f_g$ be linear polynomials in $\mathbb{C}[x_1, \ldots, x_g]$ of the form $f_i(x) = a_{i,1}x_1 + \cdots + a_{i,g}x_g$ and let V be a subvariety of $\mathbb{C}^g$ which contains no positive dimensional subvariety that is periodic under the action of $(f_1, \ldots, f_g)$ on $\mathbb{C}^g$. Then the number of points in the intersection of V and an orbit of $(f_1, \ldots, f_g)$ on $\mathbb{C}^g$ is at most $(2N)^{35N^3}$ where $N = (d+1)^g$.

Sketch of Proof – Distinct Eigenvalues

Suppose $f(x,y) = (\lambda_1 x, \lambda_2 y)$, $\mathbf{q} \in \mathbb{C}^2$, and $V = Z\left(\sum_{\substack{i+j\le d\\ i,j\ge 0}} a_{i,j}x^i y^j\right)$.

Then,

$f^n(\mathbf{q}) = (\lambda_1^n q_1, \lambda_2^n q_2)$ and if $f^n(\mathbf{q}) \in V$ then

$$\sum_{\substack{i+j\le d\\ i,j\ge 0}} a_{i,j}(\lambda_1^n q_1)^i(\lambda_2^n q_2)^j = \sum_{\substack{i+j\le d\\ i,j\ge 0}} a_{i,j} q_1^i q_2^j (\lambda_1^i \lambda_2^j)^n = 0.$$

The left-hand side expression is a polynomial-exponential sum, in the variable n, of order N where $N \le (d+1)^2$ and so there are at most $(2N)^{35N^3}$ zeroes due to Schlickewei's result.

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Suppose $f(x, y) = (\lambda x + y, \lambda y)$, $\mathbf{q} \in \mathbb{C}^2$, and $V = Z\left(\sum_{\substack{i+j\leq d \\ i,j\geq 0}} a_{i,j}x^i y^j\right)$.

Then,

$f^n(\mathbf{q}) = ((\lambda^n + n\lambda^{n-1})q_1, \lambda^n q_2)$ and if $f^n(\mathbf{q}) \in V$ then

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Suppose $f(x, y) = (ax + by, cx + dy)$, $\mathbf{q} \in \mathbb{C}^2$, and

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change coordinates so that the matrix corresponding to f is in Jordan normal form, either $M = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ or $M = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$.

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Dynamical–Mordell Lang for Linear Maps

Conjecture (D. 2012)

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Sketch of Proof

- Either directly or by induction, first show that the conjecture is true for linear maps with a nice form (those corresponding to one of the Jordan normal forms).
- Then, for any linear map, show that you may change coordinates so that the resulting matrix is in Jordan normal form.
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The dynamical Mordell–Lang conjecture for Linear Maps

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April 28, 2012

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