Mordell-Lang Questions

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e.g. let $f(x) = x^2 + 1$ and $\alpha = 0$, then

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$$\mathrm{Orb}_f(0) = \{0, 1, 2, 5, 26, 677, \ldots\};$$

- Let's suppose further that $\mathrm{Orb}_f(\alpha)$ is infinite (in dynamical terminology this means that α is not *preperiodic*).

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Let $f, g \in \mathbb{C}[x]$ each be polynomials of degree ≥ 2. Let $\alpha, \beta \in \mathbb{C}$. When can

$$\mathrm{Orb}_f(\alpha) \cap \mathrm{Orb}_g(\beta) \qquad \textit{be infinite?}$$

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Suppose that there exist positive integers m and n such that $f^m = g^n$ (we say in this case that f and g have a *common iterate*). Then, of course, we have $f^m(\alpha) = g^n(\alpha)$ so clearly the intersection $\text{Orb}_f(\alpha) \cap \text{Orb}_g(\alpha)$ is infinite for any choice of α (as long as α is not preperiodic).

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We can show this is essentially the *only way* $\mathrm{Orb}_f(\alpha) \cap \mathrm{Orb}_g(\alpha)$ can be infinite. But ruling out the case of common iterates, we have the following theorem.

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Theorem 1

(Ghioca-T-Zieve, 2008) Let $f, g \in \mathbb{C}[x]$ be polynomials of degree 2 or more. Let $\alpha, \beta \in \mathbb{C}$. If $\mathrm{Orb}_f(\alpha) \cap \mathrm{Orb}_g(\beta)$ is infinite, then there exists positive integers m and n such that $f^m = g^n$.

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The rough idea is that if there are infinitely many ℓ, and k such that $f^k(\alpha) = g^\ell(\beta)$, then for all r, s there are infinitely many integer solutions to the equation

$$f^r(x) - g^s(y) = 0$$

(we obtain these by taking $x = f^{\ell-r}(\alpha)$ and $y = g^{k-s}(\beta)$ for various k and ℓ).

Sketch of a proof of Theorem 1 (continued)

Siegel's theorem, as developed by Bilu and Tichy, says that in general equations of the form

$$P(x) - Q(y) = 0$$

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Theorem 1 is proved by showing that the *only way* there can be infinitely many solutions to

$$f^r(x) - g^s(y) = 0$$

for all r, s is to have $f^m = g^n$ for some m and n.

A geometric approach

We may think of each (f^i, g^j) as acting on $\mathbb{C} \times \mathbb{C}$ by

$$(f^i, g^j)(\alpha, \beta) = (f^i(\alpha), g^j(\beta))$$

Let Δ be the diagonal in $\mathbb{C} \times \mathbb{C}$, that is the set of all $\{(a, a) \mid a \in \mathbb{C}\}$. Then

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Thus, Theorem 1 implies that if there are infinitely many i, j such that $f^i(\alpha) = g^j(\beta)$, then there is some m, n such that $(f^m, g^n)(\Delta) = \Delta$. This gives the following reformulation of Theorem 1.

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Theorem

The set of pairs (i, j) *such that* $(f^i(\alpha), g^j(\beta)) \in \Delta$ *is a finite union of cosets of subsemigroups of* $\mathbb{N}_0 \times \mathbb{N}_0$*, where* $\mathbb{N}_0$ *is the additive semigroup of nonnegative integers.*

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Mordell-Lang theorem

This reformulation on the previous page (which motivated Theorem 1) was motivated by the so-called *Mordell-Lang* theorem of Laurent, Faltings, Vojta, and McQuillan. We state the earliest form of it, due to Laurent.

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Varieties

Definition

Let $\mathbb{P}^n_K$ be projective space over a field K. We say that Z is a projective variety in $\mathbb{P}^n_K$ if there are polynomials $F_1, \ldots, F_k \in K[x_0, \ldots, x_n]$ such that

$$Z = \{(x_0, \ldots, x_n) \in \mathbb{P}^n \quad | \quad F_1(x_0, \ldots, x_n) = \cdots = F_k(x_0, \ldots x_n) = 0\}.$$

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We say that X is a quasiprojective variety if it is the intersection of a projective variety in $\mathbb{P}^n$ with the complement of a closed projective variety in $\mathbb{P}^n$ (in other words, an open subset o a closed variety) We will drop the "quasiprojective" descriptor and just say "variety".

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Note that $\mathbb{C}^n$, for example, is a variety since it is obtained from $\mathbb{P}^n$ as the complement of they hyperplane at infinity.

Dynamical Mordell-Lang question

Question DML

Let X be a variety defined over $\mathbb{C}$, let V be a closed subvariety of X, and let S be a finitely generated commutative semigroup of maps from V to itself, and let $\alpha \in X(\mathbb{C})$. Can the set

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The answer is "no" in general, as we shall see.

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Thus, we say that the dynamical Mordell-Lang question has a negative answer for groups of additive translations.

Linear algebra

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Let S be a group of matrices in $\mathrm{GL}_n(\mathbb{C})$ and let V be one-dimensional subspace of $\mathbb{C}^n$ (i.e., a line through the origin). Then if $\alpha \in \mathbb{C}^n$ and L is the line through α in $\mathbb{C}^n$, we let S_L is the subgroup of matrices $\sigma \in S$ such that $\sigma(L) = L$.

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Moreover, it follows from Laurent's theorem that if the matrices are all simultaneously diagonalizable then the dynamical Mordell-Lang question is true, since diagonalizable matrices act like multiplicative translations – this will work for any closed subvariety V. However, when they are not diagonalizable, there are counterexamples (due to Ghioca) even for fairly simple V.

Example

Let $X = \mathbb{C}^3$, let V be the subspace given by the yz-plane, i.e. the set of all $\{0, y, z\}$, and let S be the group generated by the matrices $\begin{pmatrix} 2 & -1 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 2 \end{pmatrix}$. The dynamical Mordell-Lang question has a negative answer here.

The cyclic case

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- Φ takes the form

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  where f is an indecomposable polynomial with no periodic critical points and the subvariety V is a curve (also due to B-G-K-T).

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$\mathbb{Z}$ is in the closed unit disc in $\mathbb{C}_p$.

p-adic analytic parametrization

We show that there is a prime p and a modulus m such that for each congruence class i modulo m, there is a p-adic analytic map

$$\theta_i : \mathbb{D}_p \longrightarrow X(\mathbb{C}) \quad \text{such that} \quad \theta_i(k) = \Phi^{\ell+i+mk}(\alpha)$$

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Note that this is also a "linearizing" technique that is analogous to taking logs on a Lie group.

When does p-adic parametrization work?

We can apply the p-adic parametrization for cyclic S method whenever some iterate of α ends up in a residue class $\bar{\beta}$ modulo p such that:

- $\bar{\beta}$ is periodic modulo p (i.e., there is some power Φ^m of Φ such that Φ^m sends $\bar{\beta}$ to itself);

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Of these conditions, the one about ramification is the most serious restriction (this is why the most general case treated so far is the case where this is no ramification).

Avoiding ramification modulo p

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Question

Let $f \in \mathbb{Q}[x]$ be a polynomial of degree ≥ 2 and let $\alpha \in \mathbb{Q}$ be a point such that $\mathrm{Orb}_f(\alpha)$ does not meet the critical points of f. For what proportion of primes p is there an n such that $f^n(\alpha)$ is congruent to a critical point of f modulo p?

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We believe that the answer is "zero". In the case of quadratic polynomials, R. Jones and M. Stoll have proved this. An incomplete answer to this question gave rise to the results on maps of the form $(f, \ldots, f)$ described earlier. Getting a good answer to this question in general would be a good first step towards solving the dynamical Mordell-Lang problem in the cyclic case.

More on avoiding points modulo p

Just to make this even more concrete.

Question

Let $f \in \mathbb{Z}[x]$ be a polynomial of degree ≥ 2 and let $\alpha, \beta \in \mathbb{Z}$ be points such that there is no $n \geq 0$ for which $f^n(\alpha) = \beta$. What can be said about the set $\mathcal{S}$ primes p such that there is an n such that $f^n(\alpha) = \beta$ modulo p?

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- The only thing we can prove in general for now is that there are infinitely many primes such that there is *no n* such that $f^n(\alpha) = \beta$. In other words, there are finitely many primes p that are *not in* $\mathcal{S}$.

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- The only thing we can prove in general for now is that there are infinitely many primes such that there is *no* n such that $f^n(\alpha) = \beta$. In other words, there are finitely many primes p that are *not in* $\mathcal{S}$. (The proof of this uses Roth's theorem, which seems like overkill).

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So new ideas are needed.

Pure speculation

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One is to use the fact that the p-adic parameterizations maps convert the various elements of the semigroup into their Jacobian matrices. Then one could ask that the theorem be true when the Jacobian matrices are simultaneously diagonalizable.
That seems a bit limited, though.

More speculation

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2. The subspace $x = 0$ is stable under scalar multiplication, which does commute with the action of any matrix.

More speculation (continued)

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This may give a way towards a statement of a general dynamical Mordell-Lang theorem.