J. reine angew. Math. **706** (2015), 19–33
DOI 10.1515/crelle-2013-0060

Journal für die reine und angewandte Mathematik

Integral points in two-parameter orbits

By *Pietro Corvaja* at Udine, *Vijay Sookdeo* at Washington,
Thomas J. Tucker at Rochester and *Umberto Zannier* at Pisa

Abstract. Let K be a number field, let $f : \mathbb{P}_1 \to \mathbb{P}_1$ be a nonconstant rational map of degree greater than 1, let S be a finite set of places of K, and suppose that $u, w \in \mathbb{P}_1(K)$ are not preperiodic under f. We prove that the set of $(m, n) \in \mathbb{N}^2$ such that $f^{\circ m}(u)$ is S-integral relative to $f^{\circ n}(w)$ is finite and effectively computable. This may be thought of as a two-parameter analog of a result of Silverman on integral points in orbits of rational maps. This issue can be translated in terms of integral points on an open subset of $\mathbb{P}_1^2$; then one can apply a modern version of the method of Runge, after increasing the number of components at infinity by iterating the rational map. Alternatively, an ineffective result comes from a well-known theorem of Vojta.

1. Introduction

In 1929, Siegel [14] proved that if C is an irreducible affine curve defined over a number field K and C has at least three points at infinity, then there are at most finitely many K-rational points on C that have integral coordinates. When C has positive genus, something stronger is true: any affine curve defined over a number field has at most finitely many K-rational points that have integral coordinates. Silverman [15, Theorem A] later gave a dynamical variant of Siegel's theorem, proving that if $f : \mathbb{P}_1 \to \mathbb{P}_1$ is a rational function such that $f^{\circ 2}$ is not a polynomial and $u \in K$ is not preperiodic for f, there are only finitely many n such that $f^{\circ n}(u)$ is integral relative to the point at infinity (we will give a full definition of what it means to be integral relative to a point in Section 2). Moreover, [15, Theorem A] can be made effective, as we shall see in Section 5.

Recently, various authors (see [2,7,9]) have proposed a dynamical analog of the Mordell–Lang conjecture for semiabelian varieties. The Mordell–Lang conjecture for semiabelian varieties, which was proved by Faltings [8] and Vojta [17], states that if a finitely generated subgroup Γ of a semiabelian variety A over $\mathbb{C}$ intersects a subvariety $V \subseteq A$ in infinitely many points, then V must contain a translate of a positive-dimensional algebraic subgroup of A. One dynamical analog asserts that if one has a morphism of varieties $\Phi : X \to X$ defined over $\mathbb{C}$,

Tucker thanks the NSF for the partial support of grants 1200749 and 0854839. Corvaja and Zannier thank the ERC for the partial support of the grant "Diophantine Problems".

a subvariety $V \subseteq X$, and a point α in $X(\mathbb{C})$, then the forward orbit of α under Φ (that is, the set of distinct iterates $\Phi^{\circ n}(\alpha)$) may intersect V infinitely often only if V contains a Φ-periodic subvariety of X (that is, a subvariety W of X such that $\Phi^{\circ n}(W) = W$ for some $n > 0$) having positive dimension. Note, however, that since the forward orbit of a point under a single map is parametrized by the positive integers, it is more analogous to a cyclic group Γ than it is to an arbitrary finitely generated group Γ. Thus, one might ask for a "multi-parameter" dynamical conjecture concerning the forward orbit of α under a finitely generated semigroup of commuting maps. In [10], this problem is considered and results are obtained in the case where the subvarieties V are lines in $\mathbb{A}^2$ and the semigroup of morphisms is the set of all $(f^{\circ m}, g^{\circ n})$ for fixed polynomials f and g.

The dynamical variants of the Mordell–Lang conjecture described above all pose questions about the intersection of forward orbits with subvarieties. Here we consider the related problem of integral points in forward orbits. The main theorem of this paper is the following, which may be thought of as a two-parameter version of [15, Theorem A].

Theorem 1.1. *Let K be a number field, S a finite set of primes in K, and $f : \mathbb{P}_1 \to \mathbb{P}_1$ be a rational function with degree $d \geq 2$ that is not conjugate to a powering map $x^{\pm d}$. Let $u, w \in \mathbb{P}_1(K)$ be points that are not preperiodic for f. Then the set of $(m,n) \in \mathbb{N}^2$ such that $f^{\circ m}(u)$ is S-integral relative to $f^{\circ n}(w)$ is finite and effectively computable.*

Clearly the set (m,n) such that $f^{\circ m}(u)$ is S-integral relative to $f^{\circ n}(w)$ depends on u and w. It is possible, however, to prove an effective degeneracy result for integral points depending only on f, S, and K. This is stated in Theorem 4.1, which is phrased in terms of the S-integrality of points $(f^{\circ m}(u), f^{\circ n}(w))$ relative to inverse images of the diagonal in $\mathbb{P}_1^2$.

The outline of the paper is as follows. In Section 2, after introducing some notation, we give some equivalent notions of integrality. This will reduce our problem to the study of integral points on the complement in $\mathbb{P}_1^2$ of suitable divisors. Then, we show that a noneffective version of Theorem 1.1 can be obtained very quickly by combining [4, Appendix] with [16, Theorem 2.4.1]; this is Theorem 3.3.

In Section 4, we prove Theorem 4.1, an effective degeneracy result for K-rational points in $\mathbb{P}_1^2$ that are S-integral relative to inverse images of the diagonal under (f, f).

The technique here originates from Runge's theorem [13]; Runge treated only the case of curves, but see [6, Section 9.6] and [12] for a modern account and higher dimensional generalizations. Since inverse images of the diagonal under (f, f) have several components, one might hope to construct many rational functions ψ whose pole divisors are supported on these inverse images. The main difficulty is dealing with the points at which many components of these pole divisors meet. This can be overcome by blowing up at these points and applying a Runge-type result, Proposition 4.2, on the surface so obtained (we are indebted to the referee for suggesting this method; the alternative which we followed in a first version of this work, was more complicated); this requires showing that certain divisors in the blow-up have positive Kodaira–Iitaka dimension, which is accomplished via estimates on self-intersections.

Next, in Section 5, we use Theorem 4.1 and some simple facts about periodic curves to finish the proof of Theorem 1.1. We end with a few remarks about what happens when f is conjugate to a powering map or u or w is preperiodic.

Acknowledgement. The authors thank Aaron Levin for many helpful conversations. They are also much indebted to the referee for suggesting a more conceptual proof of Theorem 4.1.

2. Notation

Let $\mathbb{P}_n$ denote the usual projective n-space and write $\mathbb{P}_n^m$ for the m-fold Cartesian product of $\mathbb{P}_n$. We fix projective coordinates $[z_1 : \cdots : z_n]$. When convenient, we regard $\mathbb{P}_1(K)$ as $K \cup \{\infty\}$ and work in affine coordinates.

For any curve C on a surface X and any point $Q \in C(\overline{K})$, we let $m_C(Q)$ denote the *multiplicity* of C at Q; as usual (see [11, V.1], for example), the multiplicity of C at Q is defined to be the largest integer r such that $f^r \in \mathfrak{m}_Q^r$ where f is a local equation for C near Q and $\mathfrak{m}_Q$ is the maximal idea of Q in some affine neighborhood containing Q.

Let $f(x) = p(x)/q(x)$, with $p(x), q(x)$ coprime, be a rational function of degree d, and write $f^{\circ n}(x) = p_n(x)/q_n(x)$ where $p_n(x)$ and $q_n(x)$ are also coprime. The homogenization of $f^{\circ n}$ gives the rational function $F^{\circ n}([x_0 : x_1]) = [P_n(x_0, x_1) : Q_n(x_0, x_1)]$ on $\mathbb{P}_1$. Let D_n be divisor of zeros for $P_n(x_0, x_1)Q_n(y_0, y_1) - P_n(y_0, y_1)Q_n(x_0, x_1)$ and $B_i := D_i - D_{i-1}$. We note that B_i is an effective divisor. In fact, put $x = x_0/x_1$ and $y = y_0/y_1$ and let

$$\beta_i = \frac{f^{\circ i}(x) - f^{\circ i}(y)}{f^{\circ(i-1)}(x) - f^{\circ(i-1)}(y)}.$$

Then B_i is the zero divisor of β_i, in particular it is effective.

We claim that each divisor B_i has at least one component which does not belong to $D_{i-1} = B_{i-1} \cup \ldots \cup B_1$. Let us verify this fact, which is equivalent to saying that there exists a point $(x, y) \in \mathbb{P}_1 \times \mathbb{P}_1$ such that $f^{\circ(i-1)}(x) \neq f^{\circ(i-1)}(y)$ but $f^{\circ i}(x) = f^{\circ i}(y)$. Now, the rational function f, viewed as a map $\mathbb{P}_1(\overline{K}) \to \mathbb{P}_1(\overline{K})$ is surjective and not injective; take two distinct points $u, v \in \mathbb{P}_1$ with $f(u) = f(v)$ and chose $x, y \in \mathbb{P}_1$ such that $f^{\circ(i-1)}(u) = x$, $f^{\circ(i-1)}(v) = y$, and we have that $(x, y) \in B_i \setminus D_{i-1}$.

Having fixed coordinates on $\mathbb{P}_1$, we have models $(\mathbb{P}_1)_{\mathfrak{o}_K}$ and $(\mathbb{P}_1^2)_{\mathfrak{o}_K}$ for $\mathbb{P}_1$ and $\mathbb{P}_1^2$ over the ring of integers $\mathfrak{o}_K$ for a number field K. Then, for any finite set of places of K including all the archimedean places of K, we may define S-integrality in the usual ways. We say that a point $Q \in \mathbb{P}_1(K)$ is S-integral with respect to a point $P \in \mathbb{P}_1(K)$ if the Zariski closures of Q and P in $(\mathbb{P}_1)\mathfrak{o}_K$ do not meet over any primes $v \notin S$; similarly, we say that a point $Q \in \mathbb{P}_1^2(K)$ is S-integral with respect to a subvariety V of $\mathbb{P}_1^2$, defined over K, if the Zariski closures of Q and V in $(\mathbb{P}_1^2)\mathfrak{o}_K$ do not meet over any primes $v \notin S$.

Note that the diagonal D_0 in $\mathbb{P}_1^2$ is defined by the equation $x_0y_1 - y_0x_1 = 0$. So, more concretely, we say that $([a_0 : a_1], [b_0 : b_1])$ is S-integral relative to the diagonal D_0 if

$$|a_0b_1 - a_1b_0|_v \geq \max(|a_0|_v, |a_1|_v) \cdot \max(|b_0|_v, |b_1|_v)$$

for all $v \notin S$. Note that in that case the above inequality is in fact an equality. We say $[a_0 : a_1] \in \mathbb{P}_1(K)$ is S-integral relative to $[b_0 : b_1] \in \mathbb{P}_1(K)$ if $([a_0 : a_1], [b_0 : b_1])$ is S-integral relative to D_0. This definition of integrality is consistent with the previous one given that involved models for $\mathbb{P}_1$ and $\mathbb{P}_1^2$. Hence, given two points $P, Q \in \mathbb{P}_1(K)$, we see that P *is integral with respect to* Q *if and only if the pair* $(P, Q) \in \mathbb{P}_1^2(K)$ *is integral with respect to the diagonal.*

We may suppose, after enlarging our set of places S, that our rational function f has good reduction at all primes outside of S, that is that p and q have no common root at any place outside of S. Then we see that $([a_0 : a_1], [b_0 : b_1])$ is S-integral relative to D_n if

$$\begin{aligned} &|P_n(a_0, a_1) Q_n(b_0, b_1) - P_n(b_0, b_1) Q_n(a_0, a_1)|_v \\ &\quad \geq \max(|a_0|_v, |a_1|_v)^{d^n} \cdot \max(|b_0|_v, |b_1|_v)^{d^n} \end{aligned}$$

for all $v \notin S$. Note that from the definition above, it is clear that if a point is S-integral relative to D_n then it is also S-integral relative to D_m for any $m \leq n$ (as one would expect given that the support of D_m is contained in the support of D_n when $m \leq n$).

Furthermore, if S contains all the places of bad reduction for f, we have

$$\text{(2.1)} \qquad \begin{aligned} &([a_0 : a_1], [b_0 : b_1]) \text{ is } S\text{-integral relative to } D_n \\ \iff &(f^{\circ n}([a_0 : a_1]), f^{\circ n}([b_0 : b_1])) \text{ is } S\text{-integral relative to } D_0. \end{aligned}$$

We will often use coordinates (x, y) on $\mathbb{P}_1^2$ where $x = x_0/x_1$ and $y = y_0/y_1$ for projective coordinates $([x_0 : x_1], [y_0 : y_1])$. We write (∞, ∞) for the point $([1 : 0], [1 : 0])$.

We say that point z is *exceptional* for f if z is a totally ramified fixed point of $f^{\circ 2}$.

The 0-iterate $f^{\circ 0}$ is always defined to be the identity map.

For any nonconstant rational map $g : \mathbb{P}_1 \to \mathbb{P}_1$ and any point $z \in \mathbb{P}_1$, we define $e_g(z)$ to be the ramification index of z over $g(z)$. For example, if $g(x) = x^2 + 2$, then $e_g(0) = 2$ and $e_g(z) = 1$ for all $z \in \overline{K} \setminus \{0\}$.

3. Ineffective finiteness

Applying a result of Vojta, we have the following.

Theorem 3.1. *Let K be a number field, S a finite set of primes in K, and $f : \mathbb{P}_1 \to \mathbb{P}_1$ be a rational function of degree $d \geq 2$. Then the set of points in $\mathbb{P}_1^2(K)$ that are S-integral relative to D_4 lies in a proper closed subvariety Z of $\mathbb{P}_1^2$.*

Proof. The divisor D_4 has at least five irreducible components, since it contains $B_0, \dots, B_4$ and each such divisor B_i has an irreducible component not contained in the previous B_j. A theorem of Vojta [16, Theorem 2.4.1] asserts that for any divisor W on a nonsingular variety V, the points in $V(K)$ that are S-integral points relative to W are not Zariski-dense in V if W has at least $\rho + r + \dim V + 1$ components, where ρ and r are the ranks of $\mathrm{Pic}^0(V)$ and the Néron–Severi group of V, respectively. Since $\mathrm{Pic}^0(\mathbb{P}_1^2)$ is trivial and $\mathbb{P}_1^2$ has a Néron–Severi group of rank 2 (see [11, Example 6.6.1]), it follows that the set of points in $\mathbb{P}_1^2(K)$ that are S-integral relative to D_4 lies in a proper closed subvariety of $\mathbb{P}_1^2$. □

Corollary 3.2. *Let K be a number field, S a finite set of primes in K, and $f : \mathbb{P}_1 \to \mathbb{P}_1$ be a rational function with degree $d \geq 2$. There is a proper closed subvariety Y of $\mathbb{P}_1^2$ such that for any $u, w \in \mathbb{P}_1(K)$, the subvariety Y contains all but at most finitely many points $(f^{\circ m}(u), f^{\circ n}(w))$ for which $f^{\circ m}(u)$ is S-integral relative to $f^{\circ n}(w)$.*

Proof. Let Z be as in Theorem 3.1, let $\{z_1, \dots, z_e\}$ be the set of preperiodic points of f in K (note that this set must be finite), and let

$$Y = (f^{\circ 4}, f^{\circ 4})(Z) \cup \Big(\bigcup_{i=1}^{e} \mathbb{P}_1 \times \{z_i\} \Big) \cup \Big(\bigcup_{i=1}^{e} \{z_i\} \times \mathbb{P}_1 \Big). \tag{3.1}$$

If u or w is preperiodic for f, then $(f^{\circ m}(u), f^{\circ n}(w)) \in Y$ for all m, n, so we may assume that neither u nor w is preperiodic.

Now, by [15, Theorem A], for any fixed n, there are at most finitely many m such that $f^{\circ m}(u)$ is S-integral relative to $f^{\circ n}(w)$, because no $f^{\circ n}(w)$ is exceptional; likewise, there are at most finitely many n such that $f^{\circ n}(w)$ is S-integral relative to $f^{\circ m}(u)$. Thus, there are at most finitely many (m, n) with $\min(m, n) \le 4$ such that $(f^{\circ m}(u), f^{\circ n}(w))$ is S-integral relative to D_0. By (2.1), we see that if $m, n \ge 4$, then $(f^{\circ (m-4)}(u), f^{\circ (n-4)}(w))$ is S-integral relative to D_4 if and only if $(f^{\circ m}(u), f^{\circ n}(w))$ is S-integral relative to D_0. Applying Theorem 3.1, one sees then that the set of points of the form $(f^{\circ m}(w), f^{\circ n}(u))$ that are S-integral relative to D_0 is contained in Y. □

Corollary 3.3. *Let K be a number field, S a finite set of primes in K, and $f : \mathbb{P}_1 \to \mathbb{P}_1$ be a rational function with degree $d \ge 2$ that is not conjugate to a powering map $x^{\pm d}$. Let $u, w \in \mathbb{P}_1(K)$ be points that are not preperiodic for f. Then the set of $(m, n) \in \mathbb{N}^2$ such that $f^{\circ m}(u)$ is S-integral relative to $f^{\circ n}(w)$ is finite.*

Proof. Let Y be defined in equation (3.1). Applying Corollary 3.2 we obtain that $(f^{\circ m}(u), f^{\circ n}(w)) \in Y$ if $f^{\circ m}(u)$ is S-integral relative to $f^{\circ n}(w)$, apart finitely many exceptions. Since u and w are not pre-periodic, for no $(m, n) \in \mathbb{N}^2$ can happen that $(f^{\circ m}(u), f^{\circ n}(w)) \in (\bigcup_{i=1}^{e} \mathbb{P}_1 \times \{z_i\})$, so we will have $(f^{\circ m}(u), f^{\circ n}(w)) \in (f^{\circ 4}, f^{\circ 4})(Z)$. The set of points in $(f^{\circ 4}, f^{\circ 4})(Z)$ that are S-integral relative to D_0 is finite by the main theorem of [4, Appendix], since f is not conjugate to a powering map $x^{\pm d}$. Again by the fact that neither u nor w is pre-periodic, we obtain the finiteness of the possible exponents (m, n). □

4. Effective degeneracy

We will now prove the following theorem.

Theorem 4.1. *Let K be a number field and S a finite set of places of K including all the archimedean places. Let $f : \mathbb{P}_1 \to \mathbb{P}_1$ be a rational function of degree $d \ge 2$ that is not conjugate to a powering map $x^{\pm d}$. Then there is a computable integer N such that the set of points in $\mathbb{P}_1^2(K)$ that are S-integral relative to D_N lies in an effectively computable proper closed subvariety of $\mathbb{P}_1^2$.*

If f has no periodic critical points, then any point $\mathbb{P}_1^2$ is contained in at most $2d - 1$ distinct divisors B_m (see Lemma 4.3), and one may obtain a proof of Theorem 4.1 very quickly via the results of [12]. When f does have periodic critical points, however, we follow a more complicated strategy. We choose a large N (see (4.8)), resolve the embedded singularities of D_N at all of the points that are contained in more than $2d - 1$ distinct divisors B_m with

$m \leq N$ via a birational map $\pi : X \to \mathbb{P}^2_1$, and then apply a Runge-type result to $\pi^* D_N$ in X. The idea here is that resolving these singularities separates the divisors B_i over the embedded singularities. We begin by stating a Runge-type result, suggested by the referee.

4.1. A Runge-type result.

Proposition 4.2. *Let X be a nonsingular projective variety over a number field K and let S be a finite set of places of K containing the archimedean places. Let D be an effective divisor on X. Let $s = |S|$. Suppose that for any s points $P_1, \dots, P_s \in X(\overline{K})$, there exists an effective divisor E defined over K such that*

(i) $\operatorname{Supp} E \subset \operatorname{Supp} D$*;*

(ii) $\kappa(E) > 0$*, where $\kappa(E)$ is the usual Kodaira–Iitaka dimension of E (that is, the dimension of the ring $\bigoplus_{d=0}^{\infty} H^0(X, dE)$ minus one); and*

(iii) $P_1, \dots, P_s \notin \operatorname{Supp} E$.

Then the set of points of $X(K)$ that are S-integral relative to D lies in an effectively computable proper closed subset of X.

Proof. Let $\mathcal{E}$ be the set of effective reduced divisors E with $\operatorname{Supp} E \subset \operatorname{Supp} D$ and $\kappa(E) > 0$. To each $E \in \mathcal{E}$, we associate a nonconstant rational function ϕ_E such that ϕ_E has no poles outside of $\operatorname{Supp} E$; after multiplying ϕ_E by a nonzero element of K, we may assume that, for $v \notin S$, we have $|\phi_E(P)|_v \leq 1$ for all $P \in X(K)$ such that P is S-integral relative to D. Let $\Phi = \{\phi_E \mid E \in \mathcal{E}\}$.

Let v be a place of S. By [12, Lemma 2.1], there is an effectively computable constant C_v such that for any $\mathcal{E}' \subseteq \mathcal{E}$ with the property that $\bigcap_{E \in \mathcal{E}'} \operatorname{Supp} E = \emptyset$, we have

$$\min_{E \in \mathcal{E}'} |\phi_E(P)|_v \leq C_v \quad \text{for all } P \in X(K). \tag{4.1}$$

Let $P \in X(K)$ be S-integral relative to D and let

$$\mathcal{E}_{P,v} = \bigl\{E \in \mathcal{E} \mid |\phi_E(P)|_v > C_v\bigr\}.$$

Then, by equation (4.1), we have that $\bigcap_{E \in \mathcal{E}_{P,v}} \operatorname{Supp} E \neq \emptyset$. For each $v \in S$, we choose $P_v \in \bigcap_{E \in \mathcal{E}_{P,v}} \operatorname{Supp} E$. We obtain s points in this way, so our hypotheses imply that there is an $E \in \mathcal{E}$ such that none of these P_v are in $\operatorname{Supp} E$. This means that, for each $v \in S$, we have $E \notin \mathcal{E}_{P,v}$. Thus, for each $v \in S$, we have $|\phi_E(P)|_v \leq C_v$. We also have $|\phi_E(P)|_v \leq 1$ for all $v \notin S$, since P is S-integral relative to D. Thus, the height of $\phi_E(P)$ is uniformly bounded by an effective constant; this implies that $\phi_E(P)$ has only finitely many possibilities, so P lies in finitely many hypersurfaces of equation $\phi_E(P) = c$, where c belongs to a finite, effectively computable set. □

Note that in order for the method above to be truly effective, one must be able to compute the ϕ_E effectively. In our situation, this is easily done (see Remark 4.9).

Since we will apply Proposition 4.9 to strict transforms of the divisors B_i, we will need to know when these strict transforms have positive Kodaira–Iitaka dimension. The term $e_1(Q; D_i)e_2(Q; D_i)$ (see (4.3)) will provide a convenient lower bound for the self-intersection of strict transforms of the B_i.

4.2. Intersection points of B_i.

Lemma 4.3. *If the intersection of $2d$ distinct divisors $B_{m_0}, \dots, B_{m_{2d-1}}$ contains a point $p = (\xi, \eta)$, then there are distinct i and j such that $f^{\circ(m_i-1)}(\xi) = f^{\circ(m_i-1)}(\eta)$ is a periodic critical point with period dividing some $m_j - m_i$.*

Proof. First, we note that if $(c, c) \in B_1 \cap B_0$, then (c, c) has multiplicity greater than 1 on D_1. Since D_1 is defined by the equation $f(x) = f(y)$, this means that c must be a ramification point of f. Now suppose that (ξ, η) are in B_m and B_n for $m < n$. Then $(f^{\circ(n-1)}(\xi), f^{\circ(n-1)}(\eta)) \in B_1 \cap B_0$ so $f^{\circ(n-1)}(\xi) = f^{\circ(n-1)}(\eta) = c$ for c a ramification point of f. Thus, if $(\xi, \eta) \in B_{m_0} \cap \dots \cap B_{m_{2d-1}}$ for $m_0 < m_1 < \dots < m_{2d-1}$, then $f^{\circ(m_k-1)}(\xi) = f^{\circ(m_k-1)}(\eta)$ is a ramification point of f for $k = 1, \dots, 2d-1$. Since f has at most $2d-2$ ramification points, we must have $f^{\circ(m_i-1)}(\xi) = f^{\circ(m_j-1)}(\xi)$ for some $i \neq j$ with $i, j \in \{1, \dots, 2d-1\}$. □

Since f has only finitely many ramification points, we may choose an M such that the period of each periodic ramification point divides M. Note that a point x is periodic for f if and only if it is periodic for $f^{\circ M}$, since if $(f^{\circ M})^{\circ k}(x) = x$, then $f^{\circ Mk}(x) = x$ and if $f^{\circ k}(x) = x$, then $(f^{\circ M})^{\circ k}(x) = (f^{\circ k})^{\circ M}(x) = x$. Thus, every periodic ramification point of $f^{\circ M}$ is a periodic ramification point for f and is a fixed point of $f^{\circ M}$. Proving Theorem 4.1 for an iterate $f^{\circ M}$ of f is equivalent to proving it for f itself; thus, we may suppose, in view of the previous remark, that all the periodic ramification points of f are fixed points for f. After extending K, we may further assume that every fixed ramification point of f is in $\mathbb{P}_1(K)$.

Since f is not conjugate to a powering map, it has at most one exceptional point. Possibly after changing coordinates, we may assume that the exceptional point is the point at infinity. Thus, when f has an exceptional point, we will say that f is a *polynomial*. When f is not a polynomial, we may assume, possibly after changing coordinates, that no iterate $f^{\circ m}(\infty)$ is a ramification point of f.

Possibly after replacing f by $f^{\circ 2}$, we may assume that $d \geq 4$, so

$$d_i = d^i - d^{i-1} \geq \frac{3d^i}{4} \tag{4.2}$$

for all $i > 0$.

For any point $Q = (\xi_1, \xi_2) \in \mathbb{P}_1^2$ that is contained in the intersection of at least $2d$ distinct divisors B_m, we define N_Q to be the smallest non-negative integer n such that $f^n(\xi_1)$ and $f^n(\xi_2)$ are equal to the same fixed ramified point of f; that is,

$$N_Q := \min\{n \in \mathbb{N} \mid f^n(\xi_1) = f^n(\xi_2) \text{ is a fixed ramified point of } f\}.$$

Note that there may be non-periodic ramification points z such that $f^{\circ m}(z)$ is a fixed ramified point of f. When such z exist, we define M_f to be the largest integer $m \geq 1$ for which there is a non-periodic ramified point z_1 and a fixed ramified point z_2 such that $f^{\circ m}(z_1) = z_2$ and $f^{\circ(m-1)}(z_1) \neq z_2$. When no such z exist, we define M_f to be zero.

For $Q = (\xi_1, \xi_2) \in \mathbb{P}_1^2(\overline{K})$, we define

$$e_1(Q; D_i) := e_{f^{\circ i}}(\xi_1) \quad \text{and} \quad e_2(Q; D_i) := e_{f^{\circ i}}(\xi_2) \tag{4.3}$$

when $Q \in D_i$ and $e_1(Q; D_i) = e_2(Q; D_i) = 0$ for $Q \notin D_i$.

We will also use the following simple equality in various places:

$$e_{f^{\circ i}}(z) = \prod_{\ell=1}^{i} e_f\big(f^{\circ(\ell-1)}(z)\big). \tag{4.4}$$

Lemma 4.4. *Let $Q = (\xi_1, \xi_2) \in \mathbb{P}_1^2(\overline{K})$ be a point that is contained in the intersection of $2d$ distinct divisors $B_{m_0}, \dots, B_{m_{2d-1}}$. Then there are at most $M_f + 2$ divisors B_i with $i \le N_Q$ such that $Q \in B_i$.*

Proof. If $N_Q - M_f - 1 \le 0$, this is vacuously true, so suppose $N_Q - M_f - 1 \ge 0$. Let z be the fixed ramified point such that $f^{\circ N_Q}(\xi_1) = f^{\circ N_Q}(\xi_2) = z$. By the minimality of N_Q, we have $f^{\circ(N_Q-1)}(\xi_j) \ne z$ for either $j = 1$ or $j = 2$. Then $f^{\circ \ell}(\xi_j)$ is not a ramified point of f for $\ell < N_Q - M_f$ by our definition of M_f. Hence $e_j(Q; D_{N_Q-M_f-1}) \le 1$ by (4.4). This means that Q is not a singular point of $D_{N_Q-M_f-1}$; thus, Q is contained in at most one $B_i \subseteq D_{N_Q-M_f-1}$. Since there are $M_f + 1$ integers i with $N_Q - M_f \le i \le N_Q$, our proof is complete. □

Lemma 4.5. *Let $\delta > 0$. Let M_δ be a positive integer such that*

$$\Big(\frac{d-1}{d}\Big)^{M_\delta} < \frac{\delta}{2}.$$

Then, for any point $Q \ne (\infty, \infty)$ in $\mathbb{P}_1^2$ contained in the intersection of $2d$ distinct divisors $B_{m_0}, \dots, B_{m_{2d-1}}$, we have, for all $i \ge N_Q + M_\delta$,

$$[K(Q) : K]e_1(Q; D_i)e_2(Q; D_i) < \delta d_i^2. \tag{4.5}$$

Proof. Let $Q = (\xi_1, \xi_2)$. Since $f^{\circ N_Q}(\xi_1) = f^{\circ N_Q}(\xi_2) = z \in K$, we have

$$[K(\xi_j) : K]e_{f^{\circ N_Q}}(\xi_j) \le d^{N_Q} \quad \text{for } j = 1, 2.$$

Therefore,

$$[K(Q) : K]e_1(Q; D_{N_Q})e_2(Q; D_{N_Q}) \le 2d_{N_Q}^2,$$

because $(d^{N_Q})^2 \le 2d_{N_Q}^2$ by (4.2). We have $e_f(z) \le d - 1$, since z is not totally ramified. Thus, for all $i > N_Q$, we have

$$[K(Q) : K]e_1(Q; D_i)e_2(Q; D_i) \le 2\Big(\frac{d-1}{d}\Big)^{i-N_Q} d_i^2,$$

by (4.4). Therefore, we have $[K(Q) : K]e_1(Q; D_i)e_2(Q; D_i) < \delta d_i^2$ for all $i \ge N_Q + M_\delta$, as desired. □

4.3. Blowing up. This section originates from comments by the referee, whom we thank. Previously our proof was different and probably less transparent and more complicated.

As noted earlier, we wish to apply Proposition 4.2 to the pull-back of a suitable D_N under a sequence of blow-up maps. As usual, when $\pi : X \to Y$ is an onto birational map that is an isomorphism away from $W \subseteq Y$ and Z is a Zariski closed subset of X, we define the *strict transform* $\tilde{Z}$ of Z in X as the closure of $\pi^{-1}(Z \setminus (W \cap Z))$ in X. In our situation, Y will always be a nonsingular surface, Z will have dimension 1, and W will have dimension 0. For ease, when $Z \subseteq Y$ is a Zariski closed subset of dimension 1, we will call Z a curve. We call $\pi : X \to Y$ a minimal resolution of the embedded singularity of a curve $Z \subset Y$ at Q if π is obtained by repeatedly blowing up the singular points lying over Q until one obtains a nonsingular equation for $\tilde{Z}$ near the points lying over Q.

We begin with a few words on the equation defining D_i near a point $Q = (\xi_1, \xi_2)$. The curve D_i is defined by $f^{\circ i}(x) = f^{\circ i}(y)$, and $f^{\circ i}$ can be written in terms of a local parameter as

$$z_j^{e_{f^{\circ i}}(\xi_j)} + \text{ higher order terms}$$

near ξ_j for $j = 1, 2$. Thus, $f^{\circ i}$ can be written as $t_j^{e_{f^{\circ i}}(\xi_j)}$ in the completed ring $\overline{K}[[t_j]]$ near ξ_j. Therefore, any singularity on any divisor D_i is analytically isomorphic to a singularity of the form $t_1^{e_1} - t_2^{e_2} = 0$. The resolution of such singularities is simple and well known. Here we follow [11, I.4, V.3]. Blowing up, one makes the substitution $u_2 t_1 = t_2 u_1$ for projective coordinates $[u_1 : u_2]$ and obtains

$$t_1^{e_1} - t_1^{e_2}(u_2/u_1)^{e_2} \text{ for } u_1 \neq 0 \quad \text{ and } \quad t_2^{e_1}(u_1/u_2)^{e_1} - t_2^{e_2} \text{ for } u_2 \neq 0.$$

Canceling, one obtains a lower order singularity

$$t_1^{e_1-e_2} - (u_2/u_1)^{e_2} \quad \text{or} \quad (u_1/u_2)^{e_1} - t_2^{e_2-e_1}. \tag{4.6}$$

Proceeding in this way in the manner of the Euclidean algorithm, one eventually obtains the equation $z^{\gcd(e_1,e_2)} - 1$. Observe that if $\gcd(e_1, e_2) > 1$, then the embedded singularity is only resolved when the local equation takes the form $z^{\gcd(e_1,e_2)} - 1$, and that each blow-up takes place at a single singular point lying over Q.

The following lemmas are completely standard.

Lemma 4.6. *Let Z be a curve on a nonsingular surface Y, let $\pi : X \to Y$ be the blow-up of $\mathbb{P}_1^2$ at the point $Q \in Y(\overline{K})$, let m be the multiplicity of Q on Z, and let $\tilde{Z}$ be the strict transform of Z in X. Then $\tilde{Z}^2 = Z^2 - m^2$.*

Proof. The divisor $\pi^* Z$ is linearly equivalent to $\tilde{Z} + mE$ where E is the exceptional divisor for π. We have $E^2 = -1$ and $E \cdot \tilde{Z} = m$ (see [11, V.3]). Thus,

$$Z^2 = (\pi^* Z)^2 = (\tilde{Z} + mE)^2 = \tilde{Z}^2 + 2m^2 - m^2 = \tilde{Z}^2 + m^2. \qquad \square$$

Lemma 4.7. *Let Z be a curve on a nonsingular surface Y, and let $Q \in Z(\overline{K})$ be a point of Z such that the equation defining Z near Q takes the form $t_1^{e_1} - t_2^{e_2} = 0$, up to analytic isomorphism. Let $\pi_Q : X \to Y$ be a minimal resolution of Z at Q, and let $\tilde{Z}$ be the strict transform of Z in X. Then $\tilde{Z}^2 \geq Z^2 - e_1 e_2$.*

Proof. We use induction on $\max(e_1, e_2)$. If $\max(e_1, e_2) = 1$, then Z is nonsingular at Q, so no blow-up is necessary to desingularize and $Z = \tilde{Z}$. Thus, we assume without loss of generality that $\min(e_1, e_2) = e_1 > 1$. If $\pi_0 : X' \to Y$ is the blow-up of Y at Q, and $\tilde{Z}_0$ is the strict transform of Z under π_0, then $\tilde{Z}_0^2 = Z^2 - e_1^2$ by Lemma 4.6. As in (4.6), the substitution

$t_1 = t_2 u_1$ transforms the singularity into the form $(u_1/u_2)^{e_1} - t_2^{e_2-e_1}$. If $e_2 - e_1 = 0$, then the singularity is resolved, and our proof is complete since $e_1^2 \le e_1 e_2$. Otherwise, $\tilde{Z}_0$ contains a single point Q' such that $\pi_0(Q') = Q$. We may factor π_Q as $\pi_0 \circ \pi_{Q'}$; then $\tilde{Z}$ is also the strict transform of $\tilde{Z}_0$ under $\pi_{Q'}$. By induction, we have

$$\tilde{Z}^2 \ge \tilde{Z}_0^2 - e_1(e_2 - e_1) = Z^2 - e_1^2 - e_1 e_2 + e_1^2 = Z^2 - e_1 e_2. \qquad \square$$

Since each B_i is a divisor of type (d_i, d_i) on $\mathbb{P}_1^2$ (the projection on each coordinate has degree d_i), we see that $B_i^2 = 2d_i^2$ (see [11, V.2], for example). We use this, along with Lemma 4.7, to get lower bounds on the self-intersection of strict transforms of B_i under various blowing up maps.

When $Q \neq (\infty, \infty)$ or f is not a polynomial, we can expect that the product $e_1(Q; D_i)e_2(Q; D_i)$ will be small relative to d_i^2 for large i, as we saw in Proposition 4.5. When $Q = (\infty, \infty)$ and f is a polynomial, however, we have $e_1(Q; D_i)e_2(Q; D_i) = (d^i)^2$; thus, we treat this case specially in the following.

Proposition 4.8. *Let $Q_1, \dots, Q_t \in \mathbb{P}_1^2(\overline{K})$ be points that are contained in at least $2d$ distinct divisors B_m and let i be a positive integer such that $i > N_{Q_j}$ for $j = 1, \dots, t$. Let $\pi : X \to \mathbb{P}_1^2$ be the minimal resolution of the embedded singularities at all of the K-conjugates of $Q_1, \dots, Q_t$ on D_N for some $N \ge i$, and let $\tilde{B}_i$ be the strict transform of B_i in X. If*

$$\sum_{\substack{j=1 \\ Q_j \neq (\infty,\infty)}}^{t} [K(Q_j) : K] e_1(Q_j; D_i) e_2(Q_j; D_i) \le \frac{d_i^2}{9}, \tag{4.7}$$

then $\kappa(\tilde{B}_i) > 0$.

Proof. It will suffice to show that the self-intersection $\tilde{B}_i^2$ is positive, by the Riemann–Roch theorem for surfaces (see [11, V.1]). Write $Q_j = (\xi_1^{[j]}, \xi_2^{[j]})$. Since $i > N_{Q_j}$, we see that $e_1(Q_j; D_i)$ and $e_2(Q_j; D_i)$ are each divisible by $e_f(z) > 1$, where z is the ramified fixed point such that $f^{\circ N_{Q_j}}(\xi_1^{[j]}) = f^{\circ N_{Q_j}}(\xi_2^{[j]}) = z$. Thus, the embedded singularity of D_i at each conjugate of Q_j is resolved as soon as more than one point lies over Q_j in the blow-up; in particular, a minimal resolution for the embedded singularity of D_N at a conjugate of Q_j is also a minimal resolution for the embedded singularity of D_i at a conjugate of Q_j.

Now, if (∞, ∞) is not among the Q_j (which is necessarily the case when f is not a polynomial), then it follows immediately from Lemma 4.7 and (4.7) that $D_i^2 - \tilde{D}_i^2 < \frac{1}{9}(d_i)^2 < 2d_i^2$. If f is a polynomial and $(\infty, \infty) = Q_j$ for some j, then (4.7), (4.2), and Lemma 4.7 combine to give

$$D_i^2 - \tilde{D}_i^2 < (d^i)^2 + \frac{d_i^2}{9} \le \frac{16 d_i^2}{9} + \frac{d_i^2}{9} < 2d_i^2.$$

Thus, in either event, we have $D_i^2 - \tilde{D}_i^2 < 2d_i^2$. Since $D_i = B_0 + \dots + B_i$ and $\tilde{B}_\ell \cdot \tilde{B}_k \le B_\ell \cdot B_k$ for any ℓ, k, we see that

$$B_\ell^2 - \tilde{B}_\ell^2 \le D_i^2 - \tilde{D}_i^2 < 2d_i^2 \quad \text{for } \ell = 0, \dots, i.$$

Since $B_i^2 = 2d_i^2$, we must therefore have $\tilde{B}_i^2 > 0$, as desired. $\square$

4.4. Proof of Theorem 4.1. We can now prove Theorem 4.1.

Proof of Theorem 4.1. Let $s = |S|$ and let $\delta = \frac{1}{9(s+1)}$. Let

$$N = s \cdot \max(M_\delta + M_f + 2, 2d - 1) + (s + 1) \tag{4.8}$$

where M_δ is as in Lemma 4.5. Let $\pi : X \to \mathbb{P}_1^2$ be the minimal resolution of the embedded singularities of D_N at all points that are contained in at least two $2d$ distinct divisors B_i with $0 \le i \le N$.

We will show that $\pi^* D_N$ satisfies the hypotheses of Proposition 4.2, and thus that the set of points in X that are S-integral relative to $\pi^* D_N$ lies in an effectively computable proper closed subvariety of X. Now, if $P \in \mathbb{P}_1^2(K)$ is S-integral relative to D_N and $\pi(Q) = P$, then Q is S-integral relative to $\pi^* D_N$, since π is defined over K. Hence, applying Proposition 4.2 on X will complete our proof.

Let $P_1, \dots, P_s \in X(\overline{K})$. We will show that there is a divisor E with $\kappa(E) > 0$ and $\operatorname{Supp} E \subseteq \operatorname{Supp} \pi^* D_N$ such that no P_i is contained in E.

For each j, let $Q_j = \pi(P_j)$. Suppose that Q_j is in the intersection of at least $2d$ distinct divisors B_i with $0 \le i \le N$. Then by Lemma 4.4, there are at most $M_f + 2$ divisors B_i with $i \le N_Q$ such that $Q_j \in B_i$. Likewise, by Lemma 4.5, there are at most M_δ divisors B_i with $i \ge N_Q$ that fail to satisfy (4.5) with $Q = Q_j$. Thus, by (4.8) , there are at least $(s + 1)$ divisors $B_{i_0}, \dots, B_{i_s} \in \{B_0, \dots, B_N\}$ such that for all $Q_j \ne (\infty, \infty)$ we have

$$[K(Q_j) : K] e_1(Q_j; D_{i_\ell}) e_2(Q_j; D_{i_\ell}) < \delta d_{i_\ell}^2. \tag{4.9}$$

Let $\pi_1 : X_1 \to \mathbb{P}_1^2$ be the minimal resolution of the embedded singularities of D_N at all the conjugates of all the Q_j that are in the intersection of more than $2d - 1$ divisors B_i with $0 \le i \le N$. Then π factors as $\pi_1 \circ \pi_2$ for a birational map $\pi_2 : X \to X_1$. Since the strict transform $\tilde{D}_N$ is nonsingular at all points lying above any Q_j, we see that the strict transforms $\tilde{B}_{i_\ell}$ do not meet in fibers over any of the Q_j. Thus, for each P_j, there is at most one B_{i_ℓ} such that $\pi_2(P_j) \in \tilde{B}_{i_\ell}$. Since there are $s + 1$ different $\tilde{B}_{i_\ell}$ and only s different $\pi_2(P_j)$, we thus have at least one $\tilde{B}_{i_\ell}$ that does not contain any of the $\pi_2(P_j)$. We denote this divisor as $\tilde{B}_t$. We see immediately that $\pi_2^* \tilde{B}_t$ does not contain any of the P_j. Since $\kappa(\tilde{B}_t) = \kappa(\pi_2^* \tilde{B}_t)$, it will suffice to show that $\kappa(\tilde{B}_t) > 0$.

By (4.9) above, we have

$$\sum_{\substack{j=1 \\ P_j \ne (\infty,\infty)}}^{t} [K(Q_j) : K] e_1(Q_j; D_t) e_2(Q_j; D_t) \le s\delta d_i^2 \le \frac{d_i^2}{9}.$$

Thus, applying Proposition 4.8, we have $\kappa(\tilde{B}_t) > 0$, as desired. □

Remark 4.9. One can effectively compute rational functions ϕ with pole divisors contained in the support of some $\tilde{B}_t$ by constructing rational functions ϕ on $\mathbb{P}_1^2$ with pole divisors contained in B_t that satisfy certain vanishing conditions at all the conjugates of the points $Q_1, \dots, Q_s$, and pulling them back to X. Indeed, the space of rational functions on $\mathbb{P}_1^2$ of bidegree (d_t, d_t) with no poles away from B_t has dimension $2d_t^2$, while requiring that $\frac{\partial^{k+\ell}\phi}{\partial x^k \partial y^\ell}$ vanishes at Q for all $1 \le k \le e_1$ and $1 \le \ell \le e_2$ imposes $e_1 e_2$ conditions, so one can find appropriate $\phi_{\tilde{B}_t}$ whenever the conditions of Proposition 4.8 are met.

5. Effective finiteness

Silverman mentions that [15, Theorem A] can be made effective. We give a quick proof of this fact before proving Theorem 1.1.

Theorem 5.1. *Let K be a number field, S be a finite set of primes in K, and $f : \mathbb{P}_1 \to \mathbb{P}_1$ be a rational function with degree $d \geq 2$. Let a be a point that is not preperiodic for f, and let b be a point that is not exceptional for f. Then the set of n such that $f^{\circ n}(a)$ is integral relative to b is finite and effectively computable.*

Proof. Since b is not exceptional, $f^{-4}(b)$ contains at least three distinct points. To see this note that $f^{-2}(b)$ contains at least two points, since b is not exceptional. If $f^{-2}(b)$ contains exactly two points, then there is a totally ramified point in $f^{-1}(b)$ or $f^{-2}(b)$. This point cannot be fixed by f so it cannot be in both $f^{-3}(b)$ and $f^{-4}(b)$. If $f^{-3}(b)$ contains only two points, then they must both be totally ramified, so $f^{-4}(b)$ must contain a point that is not totally ramified (because f has at most two totally ramified points, by Riemann–Hurwitz), which means that $f^{-4}(b)$ contains at least three points.

For $n \geq 4$, we have that $f^{\circ n}(a)$ is S-integral relative to b if and only if $f^{\circ(n-4)}(a)$ is S-integral relative to the points in $f^{-4}(b)$. Changing coordinates, these $f^{\circ(n-4)}(a)$ are solutions to the S-unit equation, which has an effective solution (see [6, Theorem 5.4.1], for example). □

Proof of Theorem 1.1. Theorem 4.1 delivers an effectively computable one-dimensional subvariety Z such that the (m,n) with $m,n \geq N$ for which $f^{\circ m}(u)$ is S-integral relative to $f^{\circ n}(w)$ are effectively computable for all $(f^{\circ m}(u), f^{\circ n}(w))$ outside of Z.

Let c be the number of components of Z. Let $I_{u,w}$ denote the set of (m,n) such that $f^{\circ m}(u)$ is S-integral relative to $f^{\circ n}(w)$. By Theorem 5.1, we know that the set of $(m,n) \in I_{u,w}$ with $\min(m,n) \leq c + N$ is effective computable. Thus, it suffices to show that the set of $(m,n) \in I_{u,w}$ with $\min(m,n) \geq c + N$ and $(f^{\circ m}(u), f^{\circ n}(w)) \in Z$ is effective computable. Note that if $m,n \geq c \geq r$, then $f^{\circ m}(u)$ can be S-integral relative to $f^{\circ n}(w)$ only when $f^{\circ(m-r)}(u)$ is S-integral relative to $f^{\circ(n-r)}(w)$. Hence, it suffices to find all $(m,n) \in I_{u,w}$ such that $(f^{\circ m}(u), f^{\circ n}(w))$ is in $Z \cap (f,f)(Z) \cap \cdots \cap (f,f)^{\circ c}(Z)$. If this intersection is finite, we are done. Otherwise, there is a common component X among $Z, (f,f)(Z), \ldots, (f,f)^{\circ c}(Z)$. Then $(f,f)^{\circ i}(X)$ is a component of Z for $i = 0, \ldots, c$. Therefore, $(f,f)^{\circ i}(X) = (f,f)^{\circ j}(X)$ for some $c \geq j > i \geq 0$. So $(f,f)^{\circ i}(X)$ is a periodic component of Z.

Thus, we are left to show that the points of the form $(f^{\circ m}(u), f^{\circ n}(w))$ which are S-integral relative to D_0 on any periodic curve X for (f,f) can be computed. Now, since X admits a self-map of degree greater than 1, X must have genus 0 or 1. Since $X \cap D_0$ contains at least one point, we see that if X has genus 1, then the integral points on X relative to D_0 can be effectively computed (see [1,5]). If X has genus 0, and $X \cap D_0$ contains a nonexceptional point, then we are done by Theorem 5.1. If $X \cap D_0$ contains only an exceptional point z, then after changing coordinates, we may write the restriction of $(f,f)^{\circ 2}$ to X as a polynomial $P(t)$, where z is the point at infinity. If $X \cap D_0$ contains two exceptional points for $P(t)$, then, after changing coordinates, we may write the restriction of $(f,f)^{\circ 2}$ to X as a polynomial t^n (having its only pole at one exceptional point and its only zero at the other). In either case, after expanding S to a possibly larger set of primes S' we have that for any S'-integral point γ on X, each iterate $(f,f)^{\circ 2i}(\gamma)$ is S'-integral relative to D_0. This means that there are infinitely many S'-integral points relative to D_0 on X, which contradicts the main theorem of [4, Appendix]. □

6. Cyclic and exceptional cases

When f is conjugate to a powering map, we do not obtain a finiteness result. This can be seen, for example, by considering the map $f(x) = x^3$ and the points $u = 2$, $w = -2$. Then, if S is the set containing the archimedean place and the place 2, we have that $f^{\circ m}(u)$ is S-integral relative to $f^{\circ m}(w)$ for all m. On the other hand, it is possible to give a reasonable description of the (m,n) such that $f^{\circ m}(u)$ is S-integral relative to $f^{\circ n}(w)$.

In [10], it is proved that if $\deg a, \deg b > 2$ for polynomials a and b, then the set of (m,n) such that $a^{\circ m}(u) = b^{\circ n}(v)$ forms a finite union of cosets of subsemigroups of $\mathbb{N}^2$ (that is a finite union of additive translates of subsets of $\mathbb{N}^2$ that are closed under addition). Here, $\mathbb{N}$ is considered to include 0 so any finite set of (m,n) is a finite union of cosets of $(0,0)$.

For a set of places S' containing all the archimedean places, we define

$$I_{u,w,S'} = \{(m,n) \in \mathbb{N}^2 \mid f^{\circ m}(u) \text{ is } S'\text{-integral relative to } f^{\circ n}(w)\}.$$

Proposition 6.1. *Let f be conjugate to $x^{\pm d}$, let S be a finite set of places of K. Then, for some finite set of places S' with $S \subseteq S'$, the set $I_{u,w,S'}$ is a finite union of effectively computable cosets of subsemigroups of $\mathbb{N}^2$. Furthermore, the set $I_{u,w,S'}$ is finite if u and w are multiplicatively independent or if u and w are in the cyclic group generated by a non-torsion element of K^*.*

Proof. After changing coordinates by an automorphism $\sigma \in \mathrm{PGL}_2(K')$, for K' a finite extension of K, we can write $\sigma f \sigma^{-1}(x) = x^{\pm d}$. Choose a set S' of primes that includes both all the primes appearing in the coefficients or determinant of σ as well as all the primes lying over primes in S; then for any $P, Q \in \mathbb{P}_1(K')$, we have that P is S'-integral relative to Q if and only if σP is S'-integral relative to σQ (note that this choice of S' depends only on f, not on u or w). Thus, it suffices to prove the theorem when $f(x) = x^{\pm d}$.

If $f(x) = x^{-d}$, then by considering the orbit of (u,w) along with those of $(f(u), w)$, $(u, f(w))$, and $(f(u), f(w))$, we reduce to the case where $f(x) = x^{d^2}$ for some d. If u or w is zero or infinity, the conclusion is obvious. If neither u nor v is 0 or infinity, we may assume that u and w are both S'-units after expanding S'. Then $f^{\circ m}(u) - f^{\circ n}(w)$ is an S'-unit if and only if $\frac{f^{\circ m}(u)}{f^{\circ n}(w)} - 1$ is an S'-unit. Thus, if $(f^{\circ m}(u), f^{\circ n}(w))$ is S'-integral relative to D_0, then it lies on a curve of the form $x - y = \tau y$ where τ is an S'-unit such that $\tau + 1$ is also an S'-unit. By [6, Theorem 5.4.1], the set of such τ is finite and effectively computable. Thus, if u and w are multiplicatively independent, then $\frac{f^{\circ m}(u)}{f^{\circ n}(w)}$ takes on any such value τ at most once, so there are at most finitely many (m,n) such that $f^{\circ m}(u)$ is S'-integral relative to $f^{\circ n}(w)$. For any fixed value of $1 + \tau$, the set of m, n such that $u^{d^{2m}}/w^{d^{2n}} = 1 + \tau$ clearly forms a finite union of cosets of subsemigroups of $\mathbb{N}^2$.

If u and w are both in the subgroup of K^* generated by a single element z that is not a root of unity, then we may write $u = z^A$, $w = z^B$. Then, we have $z^{Ad^m - Bd^n} = (1 + \tau)$ for one of the finitely many $1+\tau$ above whenever $f^{\circ m}(u)$ is S'-integral relative to $f^{\circ n}(w)$. Now, for any constant C, the set of (m,n) such that $Ad^m - Bd^n = C$ is finite unless $C = 0$ (since $\gcd(Ad^m, Bd^n) \to \infty$ to infinity as $\min(m,n) \to \infty$), but when $C = 0$, we have $\tau = 0$, which is not an S-unit. Hence, in this case there are at most finitely many (m,n) such that $f^{\circ m}(u)$ is S'-integral relative to $f^{\circ n}(w)$. □

When at least one of u or w is preperiodic, but neither of u or w is exceptional, it is easy to see from Theorem 1.1 that the set of $(m,n) \in \mathbb{N}^2$ such that $f^{\circ m}(u)$ is S-integral relative to $f^{\circ n}(w)$ forms a finite union of effectively computable cosets of subsemigroups of $\mathbb{N}^2$. When u or w is exceptional, however, one should not expect there to be a particularly nice pattern to the set of (m,n) such that $f^{\circ m}(u)$ is S-integral relative to $f^{\circ n}(w)$. Benedetto–Briend–Perdry [3] show that if $f(x) = x^2 + \frac{x}{p}$, and v is the point at infinity, then for any set $\mathcal{U}$ of positive integers, there is a point $u \in \mathbb{Q}_p$ such that $f^{\circ m}(u) \in \mathbb{Z}_p$ if and only if $m \in \mathcal{U}$; although this is only stated over $\mathbb{Q}_p$, it is very likely that one can find examples for many complicated infinite $\mathcal{U}$ over $\mathbb{Q}$. This problem can be overcome by enlarging S to a finite set of primes S' including all the primes of bad reduction for f.

Proposition 6.2. *Suppose that w is exceptional and that there is no m such that $f^{\circ m}(u) = w$. Then for some finite set of places S' with $S \subseteq S'$, the set $I_{u,w,S'}$ is all of $\mathbb{N}^2$.*

Proof. Arguing as in Proposition 6.1, we may change coordinates so that $f^{\circ 2}$ is a polynomial and w is the point at infinity and enlarge S to some S' where our notion of S'-integrality is not affected by the coordinate change. If we enlarge S' further to include all of the places at which u, $f(u)$, or a coefficient of $f^{\circ 2}$ has a pole, then $f^{\circ 2m}(u)$ and $f^{\circ 2m}(f(u))$ are S'-integral relative to w for all m, so $I_{u,w,S'}$ is all of $\mathbb{N}^2$. □

7. Further questions

If f and g are two rational functions of degree $d > 1$ such that there are no z_1, z_2 such that $f^{\circ 2}(z_1) = g^{\circ 2}(z_2)$ with $f^{\circ 2}$ ramifying at z_1 and $g^{\circ 2}$ ramifying at z_2 (a reasonably "generic" condition), then $f^{\circ 2}(x) - g^{\circ 2}(y) = 0$ gives a nonsingular curve corresponding to a divisor D_2 of type $(2d, 2d)$ on $\mathbb{P}_1^2$. Since $d \geq 2$, we have that $D_2 + K_X$ is ample for K_X a canonical divisor of $\mathbb{P}_1^2$. Thus, Vojta's conjecture [16, Conjecture 3.4.3] would imply that the set of S-integral points relative to D_2 must be degenerate. Hence, we may expect that an analog of Theorem 3.1 holds in this case.

References

[1] *A. Baker* and *J. Coates*, Integer points on curves of genus 1, Proc. Cambridge Philos. Soc. **67** (1970), 595–602.

[2] *J. P. Bell*, A generalised Skolem-Mahler-Lech theorem for affine varieties, J. London Math. Soc. (2) **73** (2006), no. 2, 367–379.

[3] *R. Benedetto*, *J.-Y. Briend* and *H. Perdry*, Dynamique des polynômes quadratiques sur les corps locaux, J. Théor. Nombres Bordeaux **19** (2007), no. 2, 325–336.

[4] *R. L. Benedetto*, *D. Ghioca*, *P. Kurlberg*, *T. J. Tucker* and *U. Zannier*, A case of the dynamical Mordell-Lang conjecture, Math. Ann. **352** (2012), no. 1, 1–26.

[5] *Y. F. Bilu*, Quantitative Siegel's theorem for Galois coverings, Compositio Math. **106** (1997), no. 2, 125–158.
[6] *E. Bombieri* and *W. Gubler*, Heights in Diophantine geometry, New Mathematical Monographs **4**, Cambridge University Press, Cambridge 2006.
[7] *L. Denis*, Hauteurs canoniques et modules de Drinfel'd, Math. Ann. **294** (1992), no. 2, 213–223.
[8] *G. Faltings*, The general case of S. Lang's conjecture, in: Barsotti Symposium in Algebraic Geometry (Abano Terme 1991), Perspect. Math. **15**, Academic Press, San Diego (1994), 175–182.
[9] *D. Ghioca* and *T. J. Tucker*, Periodic points, linearizing maps, and the dynamical Mordell-Lang problem, J. Number Theory **129** (2009), no. 6, 1392–1403.
[10] *D. Ghioca*, *T. J. Tucker* and *M. E. Zieve*, Linear relations between polynomial orbits, Duke Math J. **161** (2012), 1379–1410.
[11] *R. Hartshorne*, Algebraic geometry, Springer, New York 1977.
[12] *A. Levin*, Variations on a theme of Runge: effective determination of integral points on certain varieties, J. Théor. Nombres Bordeaux **20** (2008), no. 2, 385–417.
[13] *C. Runge*, Über ganzzahlige Lösungen von Gleichungen zwischen zwei Veränderlichen, J. reine angew. Math. **100** (1887), 425–435.
[14] *C. L. Siegel*, Über einige Anwendungen diophantischer Approximationen, Abh. Preuss. Akad. Wiss. Phys. Math. Kl. (1929), 41–69.
[15] *J. H. Silverman*, Integer points, Diophantine approximation, and iteration of rational maps, Duke Math. J. **71** (1993), no. 3, 793–829.
[16] *P. Vojta*, Diophantine approximations and value distribution theory, Lecture Notes in Math. **1239**, Springer, Berlin 1987.
[17] *P. Vojta*, Integral points on subvarieties of semiabelian varieties. I, Invent. Math. **126** (1996), no. 1, 133–181.

Pietro Corvaja, Dipartimento di Matematica e Informatica, Università di Udine,
Via delle Scienze 206, 33100 Udine, Italy
e-mail: pietro.corvaja@uniud.it

Vijay Sookdeo, Department of Mathematics, The Catholic University of America,
Cardinal Station, Washington, DC 20064, USA
e-mail: sookdeo@cua.edu

Thomas J. Tucker, Department of Mathematics, University of Rochester,
Hylan Building, Rochester, NY 14627, USA
e-mail: ttucker@math.rochester.edu

Umberto Zannier, Scuola Normale Superiore,
Piazza dei Cavalieri 7, 56126 Pisa, Italy
e-mail: u.zannier@sns.it

Eingegangen 8. März 2012, in revidierter Fassung 17. Juni 2013