

PROOF OF A DYNAMICAL BOGOMOLOV CONJECTURE FOR LINES UNDER POLYNOMIAL ACTIONS

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ABSTRACT. We prove a dynamical version of the Bogomolov conjecture in the special case of lines in \mathbb{A}^m under the action of a map (f_1, \dots, f_m) where each f_i is a polynomial in $\overline{\mathbb{Q}}[X]$ of the same degree.

1. INTRODUCTION

In 1998, Ullmo [Ull98] and Zhang [Zha98] proved the following conjecture of Bogomolov [Bog91].

Theorem 1.1. *Let A be an abelian variety defined over a number field with Néron-Tate height \widehat{h}_{nt} and let W be a subvariety of A that is not a torsion translate of an abelian subvariety of A . Then there exists an $\epsilon > 0$ such that the set*

$$\{x \in A(\overline{\mathbb{Q}}) \mid \widehat{h}_{\text{nt}}(x) \leq \epsilon\}$$

is not Zariski dense in W .

Earlier, Zhang [Zha95a] had proved a similar result for the multiplicative group \mathbb{G}_m^n . Zhang [Zha95b, Zha06] also proposed a more general conjecture for what he called *polarizable* morphisms; a morphism $\Phi : X \rightarrow X$ on a projective variety X is said to be polarizable if there is an ample line bundle \mathcal{L} on X such that $\Phi^*\mathcal{L} \cong q\mathcal{L}$ for some integer $q > 1$. When a polarizable map Φ is defined over a number field, it gives rise to a canonical height \widehat{h}_Φ with the property that $\widehat{h}_\Phi(\Phi(\alpha)) = q\widehat{h}_\Phi(\alpha)$ for all $\alpha \in X(\overline{\mathbb{Q}})$. Zhang makes the following Bogomolov-type conjecture in this more general context.

Conjecture 1.2. *(Zhang) Let $\Phi : X \rightarrow X$ be a polarizable morphism of a projective variety defined over a number field and let W be a subvariety of X that is not preperiodic under Φ . Then there exists an $\epsilon > 0$ such that the set*

$$\{x \in W(\overline{\mathbb{Q}}) \mid h_\Phi(x) \leq \epsilon\}$$

is not Zariski dense in W .

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The definition of preperiodicity for varieties here is the same as the usual definition of preperiodicity for points. More precisely, for any quasiprojective variety X , any endomorphism $\Phi : X \rightarrow X$, and any subvariety $V \subset X$, we say that V is Φ -preperiodic if there exists $N \geq 0$, and $k \geq 1$ such that $\Phi^N(V) = \Phi^{N+k}(V)$. Note that when A is an abelian variety and Φ is a multiplication-by- n map, a subvariety W is preperiodic if and only if it is a torsion translate of an abelian subvariety of A .

In this paper, we prove the following special case of Conjecture 1.2.

Theorem 1.3. *Let $f_1, \dots, f_m \in \overline{\mathbb{Q}}[X]$ be polynomials of degree $d > 1$, let $\Phi := (f_1, \dots, f_m)$ be their coordinatewise action on \mathbb{A}^m , and let L be a line in \mathbb{A}^m defined over $\overline{\mathbb{Q}}$. If L is not Φ -preperiodic, then there exists an $\epsilon > 0$ such that*

$$S_{L, \Phi, \epsilon} := \{x \in L(\overline{\mathbb{Q}}) \mid \widehat{h}_\Phi(x) \leq \epsilon\}$$

is finite.

Baker and Hsia [BH05, Theorem 8.10] previously proved Theorem 1.3 in the special case where $f_1 = f_2$ and $m = 2$.

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2. PRELIMINARIES

Heights. Let $\mathbb{M}_{\mathbb{Q}}$ be the usual set of absolute values on \mathbb{Q} , normalized so that the archimedean absolute value is simply the absolute value $|\cdot|$ and $|p|_p = 1/p$ for each p -adic absolute value $|\cdot|_p$. For any extension K of \mathbb{Q} we define \mathbb{M}_K to be the set of absolute values on K that extend elements of $\mathbb{M}_{\mathbb{Q}}$. Then, for any $x \in \overline{\mathbb{Q}}$ we define the Weil height of x to be

$$h(x) = \frac{1}{[\mathbb{Q}(x) : \mathbb{Q}]} \cdot \sum_{v \in \mathbb{M}_{\mathbb{Q}(x)}} \sum_{\substack{w|v \\ w \in \mathbb{M}_{\mathbb{Q}(x)}}} \log \max\{|x|_w^{[\mathbb{Q}(x)_w : \mathbb{Q}_v]}, 1\}$$

where \mathbb{Q}_v and $\mathbb{Q}(x)_w$ are the completions of \mathbb{Q} and $\mathbb{Q}(x)$ at v and w respectively (see [BG06, Chapter 1] for details).

For a polynomial $f \in \overline{\mathbb{Q}}[X]$ of degree greater than 1, define the f -canonical height $\widehat{h}_f : \overline{\mathbb{Q}} \rightarrow \mathbb{R}_{\geq 0}$ by

$$(2.1) \quad \widehat{h}_f(x) = \lim_{k \rightarrow \infty} \frac{h(x)}{(\deg f)^k},$$

following Call-Silverman [CS93].

Let $f_1, \dots, f_m \in \overline{\mathbb{Q}}[X]$ be polynomials of degree $d > 1$, and let $\Phi := (f_1, \dots, f_m)$ be their coordinatewise action on \mathbb{A}^m ; that is,

$$\Phi(x_1, \dots, x_m) = (f_1(x_1), \dots, f_m(x_m)).$$

We define the Φ -canonical height $\widehat{h}_\Phi : \mathbb{A}^m(\overline{\mathbb{Q}}) \rightarrow \mathbb{R}_{\geq 0}$ by

$$\widehat{h}_\Phi(x_1, \dots, x_m) = \sum_{i=1}^m \widehat{h}_{f_i}(x_i).$$

Note that while \mathbb{A}^m is not a projective variety, Φ extends to a map $\tilde{\Phi} : (\mathbb{P}^1)^m \rightarrow (\mathbb{P}^1)^m$. Furthermore, $\tilde{\Phi}$ is polarizable, since

$$\tilde{\Phi}^* \sum_{i=1}^m \text{pr}_i^* \mathcal{O}_{\mathbb{P}^1}(1) \cong \sum_{i=1}^m \text{pr}_i^* \mathcal{O}_{\mathbb{P}^1}(d),$$

where pr_i is the projection of $(\mathbb{P}^1)^m$ onto its i -th coordinate.

Remark. Theorem 1.3 is *not* true if one allows the polynomials f_i to have different degrees. This is easily seen, for example, in the case where $m = 2$, the line L is the diagonal, and $f_2 = f_1^2$. The map $\Phi = (f_1, \dots, f_m)$ is only polarizable when $\deg f_i = \deg f_j$, so this is not a counterexample to Conjecture 1.2.

Julia sets. For a polynomial $g \in \mathbb{C}[X]$, we let $J(g)$ denote the Julia set of g . See [Bea91, Chapter 3] or [Mil99] for the definition of a Julia set of a rational function over the complex numbers.

3. PROOF OF OUR MAIN RESULT

Proof of Theorem 1.3. Suppose that for every $\epsilon > 0$, the set $S_{L, \Phi, \epsilon}$ is infinite. We will show that this implies that L must be Φ -preperiodic.

We first note that it suffices to prove the theorem for the line $L' = (\sigma_1, \dots, \sigma_m)(L)$ and the map

$$\Phi' = (\sigma_1 f_1 \sigma_1^{-1}, \dots, \sigma_m f_m \sigma_m^{-1})$$

for some linear automorphisms $\sigma_1, \dots, \sigma_m$ of \mathbb{A}^1 . This follows from the fact that L is preperiodic for Φ if and only if L' is preperiodic for Φ' along with the equality

$$(3.1) \quad \widehat{h}_{\Phi'}(\sigma_1 \alpha_1, \dots, \sigma_m \alpha_m) = \widehat{h}_\Phi(\alpha_1, \dots, \alpha_m).$$

Note that (3.1) is a simple consequence of Definition 2.1, since $|h(\sigma_i x) - h(x)|$ is bounded for all $x \in \overline{\mathbb{Q}}$.

We now proceed by induction on m ; the case $m = 1$ is obvious.

If the projection of L on any of the coordinates consists of only one point, we are done by the inductive hypothesis. Indeed, without loss of generality, assume the projection of L on the first coordinate equals $\{z_1\}$, then $L = \{z_1\} \times L_1$, where $L_1 \subset \mathbb{A}^{m-1}$ is a line, and $\widehat{h}_{f_1}(z_1) = 0$. Since only preperiodic points have canonical height equal to 0 (see [CS93, Cor. 1.1.1]), we conclude that z_1 is f_1 -preperiodic, and thus we are done by the induction hypothesis applied to L_1 .

Suppose now that L projects dominantly onto each coordinate of \mathbb{A}^m . For each $i = 2, \dots, m$, we let L_i be the projection of L on the first and the i -th

coordinates of \mathbb{A}^m . Then L_i is a line given by an equation $X_1 = \sigma_i(X_i)$, for some linear polynomial $\sigma_i \in \overline{\mathbb{Q}}[X]$. Clearly, it suffices to show that for each $i = 2, \dots, m$, the line L_i is preperiodic under the action of (f_1, f_i) on the corresponding two coordinates of \mathbb{A}^m .

Let $\tilde{f}_i := \sigma_i f_i \sigma_i^{-1}$ and let $\Delta = (x, x) \in \mathbb{A}^2$ be the diagonal on \mathbb{A}^2 . By our remarks at the beginning of the proof, it suffices to show that $(\text{id}, \sigma_i)(L_i) = \Delta$ is preperiodic under the action of (f_1, \tilde{f}_i) . Furthermore, the fact that we have an infinite sequence $(z_{n,1}, z_{n,i}) \in L_i(\mathbb{Q})$ with

$$\lim_{n \rightarrow \infty} \widehat{h}_{f_1}(z_{n,1}) = \lim_{n \rightarrow \infty} \widehat{h}_{\tilde{f}_i}(z_{n,i}) = 0$$

means that we have

$$\lim_{n \rightarrow \infty} \widehat{h}_{\tilde{f}_i}(z_{n,1}) = 0,$$

because of (3.1). Fix an embedding $\theta : \overline{\mathbb{Q}} \rightarrow \mathbb{C}$ and let f_1^θ and \tilde{f}_i^θ be the images of f_1 and \tilde{f}_i , respectively, in $\mathbb{C}[X]$ under this embedding. Then, by [BH05, Corollary 4.6], the Galois orbits of the points $\{z_{n,1}\}_{n \in \mathbb{N}}$ are equidistributed with respect to the equilibrium measures on the Julia sets of both f_1^θ and \tilde{f}_i^θ . Since the support of the equilibrium measure μ_g of a polynomial $g \in \mathbb{C}[X]$ is equal to the Julia set of g ([BH05, Section 4]), we must have $J(\tilde{f}_i^\theta) = J(f_1^\theta)$.

By [Bea92, Theorem 1] (see also [BE87, AH96]), there exists a conformal Euclidean symmetry $\mu_i : z \rightarrow a_i z + b_i$ such that $\mu_i(J(f_1^\theta)) = J(f_1^\theta)$ and $\tilde{f}_i^\theta = \mu_i f_1^\theta$. Note that a_i and b_i must be in the image of $\overline{\mathbb{Q}}$ under θ since the coefficients of f_1^θ and \tilde{f}_i^θ are. Let τ_i be the map $\tau_i : z \rightarrow \theta^{-1}(a_i)z + \theta^{-1}(b_i)$. Then we have $f_i = \tau_i f_1$.

If τ_i has infinite order, then it follows from [Bea90, Lemma 4] that there exist linear polynomials γ_1, γ_i such that $\gamma_1 f_1 \gamma_1^{-1} = \gamma_i \tilde{f}_i \gamma_i^{-1} = X^d$. In this case, we reduce our problem to the usual Bogomolov conjecture for \mathbb{G}_m^2 , proved by Zhang [Zha92]. Indeed, Zhang proves that if a curve C in \mathbb{G}_m^2 has an infinite family of algebraic points with height tending to zero, then it must be a torsion translate of an algebraic subgroup of \mathbb{G}_m^2 ; that is, $C = \xi A$ where ξ has finite order and A is an algebraic subgroup of \mathbb{G}_m^2 . Since $(\xi A)^n = \xi^n A$ and ξ has finite order, it is clear that such a curve is preperiodic under the map $(X, Y) \mapsto (X^d, Y^d)$.

We may suppose then that τ_i has finite order $\ell \geq 1$. By [Bea90, Lemma 7], we have $f_1 \tau_i = \tau_i^\ell f_1$. Thus, we have

$$\tilde{f}_i^k = \tau_i^{(d^k - 1)/(d - 1)} f_1^k$$

for all $k \geq 1$. Since τ_i has finite order, we conclude that the set

$$\{\tau_i^{(d^k - 1)/(d - 1)}\}_{k \geq 0}$$

is finite. This implies that the set of curves of the form $(f_1^k, \tilde{f}_i^k)(\Delta)$ is finite, which means the diagonal subvariety Δ is preperiodic under the action of (f_1, \tilde{f}_i) . \square

We believe that it is possible to extend the methods of the proof of Theorem 1.3 to the case of arbitrary rational maps $\varphi_1, \dots, \varphi_m$ of the same degree, though the proof seems to be much more difficult, requiring in particular Mimar's [Mim97] results on arithmetic intersections of metrized line bundles and an analysis of Douady-Hubbard-Thurston's [DH93] classification of critically finite rational maps with parabolic orbifolds. We intend to treat this problem in a future paper.

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