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Applications of p-Adic Analysis for Bounding Periods of Subvarieties Under Étale Maps

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Using methods of p-adic analysis, we obtain effective bounds for the length of the orbit of a preperiodic subvariety $Y \subset X$ under the action of an étale endomorphism of X. As a corollary of our result, we obtain effective bounds for the size of torsion of any semi-abelian variety over a finitely generated field of characteristic 0. Our method allows us to show that any finitely generated torsion subgroup of $\mathrm{Aut}(X)$ is finite. This yields a different proof of Burnside's problem for automorphisms of quasiprojective varieties X defined over a field of characteristic 0.

1 Introduction

In [27], Morton and Silverman conjecture that there is a constant $C(N, d, D)$ such that for any morphism $f: \mathbb{P}^N \longrightarrow \mathbb{P}^N$ of degree d defined over a number field K with $[K:\mathbb{Q}] \leq D$, the number of preperiodic points of f in $\mathbb{P}^N(K)$ is $\leq C(N, d, D)$. This conjecture remains very much open, but in the case where f has good reduction at a prime $\mathfrak{p}$, a great deal has been proved about bounds depending on $\mathfrak{p}$, N, d, D (see [18, 28, 36]).

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In this paper, we study the more general problem of bounding periods of subvarieties of any dimension. We prove the following theorem.

Theorem 1.1. Let K be a finite extension of $\mathbb{Q}_p$, let $\mathfrak{o}_v$ be the ring of integers of K, let k_v be its residue field and let e be the ramification index of $K/\mathbb{Q}_p$. Let $\mathcal{X}$ be a smooth $\mathfrak{o}_v$-scheme whose generic fiber X has dimension g, let $\Phi : \mathcal{X} \longrightarrow \mathcal{X}$ be étale, let $\mathcal{Y}$ be a subvariety of $\mathcal{X}$, and assume there is a point on $\mathcal{Y}(\mathfrak{o}_v)$ which is smooth on the generic fiber of $\mathcal{Y}$. We let $\bar{\mathcal{X}}$ be the special fiber of $\mathcal{X}$, and let r be the smallest non-negative integer larger than $(\log(e) - \log(p-1))/\log(2)$. If $\mathcal{Y}$ is preperiodic under the action of Φ, then the length of its orbit is bounded above by $p^{1+r} \cdot \#\mathrm{GL}_g(k_v) \cdot \#\bar{\mathcal{X}}(k_v)$. □

Theorem 1.1 is proved using p-adic analytic parameterization of forward orbits under the action of an étale endomorphism of a quasiprojective variety. The same method can be used to study finitely generated torsion subgroups of $\mathrm{Aut}(X)$, when X is a quasi-projective variety defined over a field K of characteristic zero. Theorem 3.1 gives an upper bound on the size of the largest finitely generated torsion subgroup in $\mathrm{Aut}X$ when X has a smooth model over a finite extension of the p-adic integers; the bound depends only on the dimension of X and on the number of points in the special fiber of this model. This gives rise to a new proof of the following theorem of Bass and Lubotzky [4].

Theorem 1.2. Let X be a geometrically irreducible quasiprojective variety defined over a field of characteristic 0. Then each finitely generated torsion subgroup H of $\mathrm{Aut}(X)$ is finite. □

Theorem 1.2 shows, in particular, that the Burnside problem has an affirmative solution for automorphism groups of quasiprojective varieties. We recall that the Burnside problem is said to have a positive answer for a group G if every finitely generated torsion subgroup of G is finite. The first substantial result in this area was due to Burnside (cf. [21, Section 9]), who showed that if H is a (not necessarily finitely generated) torsion subgroup of $\mathrm{GL}_n(\mathbb{C})$ of exponent d, then the order of H could be bounded in terms of d and n. Using a specialization argument and applying Burnside's result, Schur [31] later showed that every finitely generated torsion subgroup of $\mathrm{GL}_n(\mathbb{C})$ is finite. Proofs of geometric Burnside-type results generally proceed along similar lines as that of the Burnside–Schur theorem: one first uses specialization to reduce to the case that the base field is a finitely generated extension of the prime field; one then shows that in this case a torsion subgroup necessarily has bounded exponent and is finite. An interesting problem that arises naturally is to then bound the exponent and the order of a torsion subgroup of Aut(X) in terms of geometric data and the field k of definition for X when k is a finitely generated extension of $\mathbb{Q}$. Some work in this direction has been done by Serre [32], who gave sharp upper-bounds on the sizes of torsion subgroups of the group of birational transformations of $\mathbb{P}^2(k)$ when k is a finitely generated extension of $\mathbb{Q}$.

The p-adic analytic parameterization of forward orbits under the action of an automorphism of a quasiprojective variety can be used in different directions as well. We prove the following result.

Theorem 1.3. Let X be an irreducible quasiprojective variety defined over $\bar{\mathbb{Q}}$ of dimension larger than 1, and let Φ be an automorphism of X for which there exists no nonconstant $f \in \bar{\mathbb{Q}}(X)$ such that $f \circ \Phi = f$. Then, there exists a codimension-2 subvariety Y whose orbit under Φ is Zariski dense in X. □

For any subvariety Y, its orbit under Φ is the union (denoted $\mathcal{O}_{\Phi}(Y)$) of all $\Phi^n(Y)$ for $n \in \mathbb{N}$.

We note that if there exists a nonconstant $f \in \bar{\mathbb{Q}}(X)$ such that $f \circ \Phi = f$, then for any subvariety $Y \subseteq X$, we have that $f(\mathcal{O}_{\Phi}(Y)) = f(Y)$; so, the orbit of Y would not be Zariski dense in X since it would be contained in a smaller dimension subvariety (note that Zariski closure of $f(Y)$ has dimension bounded above by $\dim(Y) < \dim(X)$).

Theorem 1.3 yields positive evidence to two conjectures in arithmetic geometry. On one hand, we have the *potential density question*, that is, describe the class of varieties X defined over $\bar{\mathbb{Q}}$ for which there exists a number field K such that $X(K)$ is Zariski dense in X (see [2, 9, 17] for various results regarding potentially dense varieties). Our Theorem 1.3 yields that if Y is potentially dense, then X is potentially dense. In particular, if X is a surface, then Theorem 1.3 yields the existence of a point $x \in X(\bar{\mathbb{Q}})$ whose orbit is Zariski dense in X (note that in this case, Y is a finite collection of points and since X is irreducible, we obtain that a single orbit under Φ must be Zariski dense in X). Hence, Theorem 1.3 yields a positive answer for automorphisms of surfaces for the following conjecture (proposed independently by Amerik et al. [2], and Medvedev and Scanlon [25]).

Conjecture 1.4. Let X be a quasiprojective variety defined over an algebraically closed field K of characteristic 0. Let $\Phi : X \longrightarrow X$ be an endomorphism defined over K such that there exists no positive-dimensional variety Y and no dominant rational map $\Psi : X \longrightarrow Y$ such that $\Psi \circ \Phi = \Psi$ generically. Then, there exists $x \in X(K)$ such that $\mathcal{O}_{\Phi}(x)$ is Zariski dense in X. □

Alternatively, one can formulate the hypothesis in Conjecture 1.4 by asking that Φ does not preserve a rational fibration, that is, there exists no nonconstant $f \in K(X)$ such that $f \circ \Phi = f$. Arguing as above, if Φ preserves a nonconstant rational fibration f, then there is no point x with Zariski dense orbit under Φ. Indeed, otherwise the entire orbit of x would be mapped by f into $f(x)$; hence, $\mathcal{O}_\Phi(x)$ cannot be Zariski dense in X because this would imply that f is constant. Conjecture 1.4 strengthens a conjecture of Zhang, which was also the motivation for both Amerik, Bogomolov, and Rovinsky, respectively for Medvedev and Scanlon for formulating the above Conjecture 1.4. Motivated by the dynamics of endomorphisms on abelian varieties, Zhang [35] proposed the following question for polarizable endomorphisms. We say that Φ is *polarizable* if there exists an ample line bundle $\mathcal{L}$ on X such that $\Phi^*(\mathcal{L}) = \mathcal{L}^{\otimes d}$ (in $\mathrm{Pic}(X)$) for some integer $d > 1$. Zhang [35] conjectured that given a projective variety X defined over a number field K endowed with a polarizable endomorphism Φ, then there exists a point $x \in X(\bar{K})$ whose orbit under Φ is Zariski dense in X.

In [3], Amerik and Campana prove Conjecture 1.4 for projective varieties of trivial canonical bundle defined over an uncountable field K. However, if $K = \bar{\mathbb{Q}}$, then the problem seems much more difficult. Only recently, Medvedev and Scanlon [25] proved Conjecture 1.4 when $\Phi = (f_1, \ldots, f_N)$ is an endomorphism of $\mathbb{A}^N$ given by N one-variable polynomials f_i defined over $\bar{\mathbb{Q}}$. Also, Junyi [19, Theorem 1.4] proved Conjecture 1.4 for birational maps on projective surfaces. Finally, connected to Conjecture 1.4, we mention Amerik's result [1] who proved (using the p-adic approach introduced in [7]) that most orbits of algebraic points are infinite under the action of an arbitrary rational self-map (of infinite order).

Our proof of Theorem 1.3 uses a result (see Theorem 4.2) that gives an upper bound for the period of codimension-1 subvarieties of X which are periodic under Φ; in particular, this yields that the union of all periodic hypersurfaces is Zariski closed. We note that Cantat [10, Theorem A] proved a similar bound for the number of periodic hypersurfaces under the stronger hypothesis that there exist no nonconstant rational function f and no constant α such that $f \circ \Phi = \alpha \cdot f$. Our Theorem 1.1 yields that each periodic subvariety with a point over some complete v-adic field has bounded period. So, if Y is a codimension-2 subvariety of X which is neither periodic, nor contained in one of the finitely many codimension-1 periodic subvarieties, then its orbit under Φ is Zariski dense. Using the same approach, it is immediate to get the existence of codimension-1 subvarieties with a Zariski dense orbit in X.

Using the hypothesis that X contains a Zariski dense orbit, and also using Vojta's proof of the Mordell–Lang Theorem for semiabelian varieties (see [34]) we obtain the following stronger bound for the number and period of codimension-1 periodic subvarieties.

Theorem 1.5. Let X be a quasi-projective variety and let $\sigma : X \longrightarrow X$ be an automorphism defined over a number field K and suppose that there is a point $x \in X(K)$ such that the orbit of x under σ is Zariski dense in X. Let Y be a projective closure of X, and let $\rho(Y)$ be the Picard number of Y. Then, any σ-invariant closed subset W of X has at most $\dim X - h^1(Y, \mathcal{O}_Y) + \rho(Y)$ geometric components of codimension 1. □

Of course, Conjecture 1.4 for an automorphism $\sigma : X \longrightarrow X$ follows immediately whenever one knows that the union of all σ-invariant subvarieties is Zariski closed. Hence, the following equivalence is of interest here.

Definition 1.6. Let X be a quasi-projective variety over a field K and let $\sigma : X \to X$ be an automorphism of X. We say that (X, σ) satisfies the *geometric Dixmier–Moeglin equivalence* if the following are equivalent for each σ-stable subvariety Y of X:

(1) there exists a point $y \in Y$ such that $\{\sigma^n(y) : n \in \mathbb{Z}\}$ is Zariski dense in Y;
(2) the union of all proper σ-invariant subvarieties of Y is Zariski closed;
(3) there does not exist a nonconstant $f \in k(Y)$ such that $f \circ \sigma = f$. □

We note that the geometric Dixmier–Moeglin equivalence does not hold in general—for example, there are Hénon maps of $\mathbb{A}^2$ with the property that (3) holds but (2) does not (cf. Devaney and Nitecki [14] and Bedford and Smillie [5, Theorem 1])—but it is conjectured to hold when X is smooth and projective and σ has zero entropy. As before, for X a complex variety, we have the implications (2) $\Longrightarrow$ (1) $\Longrightarrow$ (3) [8]. Theorem 1.3 proves that the equivalences from Definition 1.6 hold for any surface.

Here is the plan of our paper: in Section 2, we prove some preliminary results (see Proposition 2.1) for rigid analytic functions, which we use then in Section 3 for proving Theorems 1.1 and 1.2 and their corollaries. In Section 4, we find an upper bound for the period of codimension-1 periodic subvarieties under the action of an automorphism Φ of a quasiprojective variety which does not preserve a rational fibration (see Theorem 4.2). In Section 5, using Theorem 4.2, we prove Theorem 1.3 and Theorem 1.5. Finally, we conclude with Section 6 in which we discuss related questions (in the spirit of Poonen's conjectures [29]) about uniform boundedness for periods of points in algebraic families of endomorphisms.

2 Nonarchimedean Analysis

2.1 Power series

The setup for this section is as follows: p is a prime number, $K_v/\mathbb{Q}_p$ is a finite extension, while the v-adic norm $|\cdot|_v$ satisfies $|p|_v = 1/p = |p|_p^{1/e}$ (i.e., e is the ramification index for this extension). We let $\mathfrak{o}_v$ be the ring of v-adic integers of K_v, let π be a uniformizer of $\mathfrak{o}_v$, and we let k_v be its residue field.

We let g be a positive integer, and let c be a positive real number. For two power series $F, G \in \mathfrak{o}_v[\![z_1, \ldots, z_g]\!]$, we write $F \equiv G \pmod{p^c}$ if each coefficient a_α of $F - G$ satisfies $|a_\alpha|_v \leq |p|_v^c$. Alternatively, for some $m \in \mathbb{N}$, we use the notation $F \equiv G \pmod{\pi^m}$ if $F - G \in \pi^m \mathfrak{o}_v[\![z_1, \ldots, z_g]\!]$. More generally, for g-tuples of power series $\mathcal{F} := (F_1, \ldots, F_g)$ and $\mathcal{G} := (G_1, \ldots, G_g)$ we write $\mathcal{F} \equiv \mathcal{G} \pmod{p^c}$ if $F_i \equiv G_i \pmod{p^c}$ for each i; similarly, $\mathcal{F} \equiv \mathcal{G} \pmod{\pi^m}$ if $F_i \equiv G_i \pmod{\pi^m}$ for each i. Finally, for each $n \in \mathbb{N}$, we denote by $\mathcal{F}^n$ the composition of $\mathcal{F}$ with itself n times. We note that in general, the composition $\mathcal{F}^n$ may not be well-defined; however, it is well defined in the following special case: there exists another g-tuple of power series $\mathcal{H} := (H_1, \ldots, H_g)$ such that

(1) each $H_i \in \mathfrak{o}_v[\![z_1, \ldots, z_g]\!]$; and
(2) $F_i = \frac{1}{\pi} \cdot H_i(\pi z_1, \ldots, \pi z_g)$ for each $i = 1, \ldots, g$.

Essentially, the above conditions yield that the coefficients of each F_i converge sufficiently fast to 0 so that the composition $\mathcal{F}^n$ is well defined.

We use the following result in Section 3.

Proposition 2.1. Let $C \in \mathfrak{o}_v^g$, let $L \in \mathrm{GL}_g(\mathfrak{o}_v)$, and let $F_1, \ldots, F_g \in \mathfrak{o}_v[\![z_1, \ldots, z_g]\!]$, such that for $z := (z_1, \ldots, z_g)$, we have

$$\mathcal{F}(z) := (F_1, \ldots, F_g)(z) \equiv C + Lz \pmod{\pi}.$$

Let $m = p^{1+r} \cdot \#\mathrm{GL}_g(k_v)$ where r is any nonnegative integer larger than $(\log(e) - \log(p-1))/\log(2)$. Then, $\mathcal{F}^m(z) \equiv z \pmod{p^c}$ for some $c > 1/(p-1)$. □

Proof. Let $s := \#\mathrm{GL}_g(k_v)$; then letting id be the identity g-by-g matrix, we get $L^s \equiv \mathrm{id} \pmod{\pi}$ (since $L \in \mathrm{GL}_g(\mathfrak{o}_v)$). Thus, there exists some $D \in \mathfrak{o}_v^g$ such that

$$\mathcal{F}^s(z) \equiv D + L^s z \equiv D + z \pmod{\pi}.$$

Then, $\mathcal{F}^{ps} \equiv z \pmod{\pi}$. Hence, we are left to show that if $\mathcal{F}(z) \equiv z \pmod{\pi}$ and if r is the least nonnegative integer $> (\log(e) - \log(p-1))/\log(2)$, then $\mathcal{F}^{p^r}(z) \equiv z \pmod{p^c}$ for some $c > 1/(p-1)$. Clearly, if $\mathcal{G}(z) \equiv z \pmod{p^c}$, then also $\mathcal{G}^{p^k}(z) \equiv z \pmod{p^c}$ for any positive integer k.

If $e < p-1$, then $r = 0$ works since $|\pi|_v = |p|_v^{1/e} = p^{-1/e} < p^{-1/(p-1)}$. So, from now on, we assume $e \geq p-1$.

We let $\mathcal{F}(z) = z + \mathcal{H}(z)$, where each coefficient of $\mathcal{H}$ is in $\pi \cdot \mathfrak{o}_v$. Then, $\mathcal{F}^p(z) = z + p\mathcal{H}(z) + \mathcal{H}_1(z)$, where $\mathcal{H}_1 \equiv 0 \pmod{\pi^2}$. Thus, $\mathcal{F}^p(z) \equiv z \pmod{\pi^2}$. By induction, we obtain that

$$\mathcal{F}^{p^r}(z) \equiv z \pmod{\pi^{\min\{e+1, 2^r\}}}.$$

So, if $r > (\log(e) - \log(p-1))/\log(2)$, then $|\pi|_v^{2^r} = p^{-\frac{2^r}{e}} < p^{-\frac{1}{p-1}}$, while $|\pi|_v^{e+1} < |p|_v \leq p^{-\frac{1}{p-1}}$, and so indeed

$$\mathcal{F}^{p^r} \equiv z \pmod{p^c} \text{ for some } c > \frac{1}{p-1},$$

which yields the desired conclusion. ■

2.2 Algebraic geometry

We need the following application of the implicit function theorem on Banach spaces.

Proposition 2.2. Let $(K_v, |\cdot|_v)$ be a finite extension of $\mathbb{Q}_p$ with residue field k_v, and let $\mathfrak{o}_v$ be the ring of v-adic integers of K_v. Let X be a quasiprojective variety defined over K_v, let $\mathcal{X}$ be a $\mathfrak{o}_v$-scheme whose generic fiber is isomorphic to X, let $r : \mathcal{X}(K_v) \longrightarrow \bar{\mathcal{X}}(k_v)$ be the usual reduction map to the special fiber $\bar{\mathcal{X}}$ of $\mathcal{X}$, and let $\iota : \mathcal{X}(\mathfrak{o}_v) \longrightarrow X(K_v)$ be the usual map coming from base extension. Let $\alpha \in \mathcal{X}(\mathfrak{o}_v)$ such that $\iota(\alpha)$ is a smooth point on X and let $U_{\bar{\alpha}} = \{\beta \in \mathcal{X}(\mathfrak{o}_v) : r(\alpha) = r(\beta)\}$ and let $U = \iota(U_{\bar{\alpha}})$. Then, U is Zariski dense in X. □

Proof. Let $x = \iota(\alpha)$. We consider an affine chart containing the point $x \in X$ after viewing X as a subset of the n-dimensional projective space defined over K_v. So, letting $d = \dim(X)$, then there exist $(n-d)$ polynomials f_i, which we may suppose are defined over $\mathfrak{o}_v$ in n variables $z_1, \ldots, z_n$ such that locally at x the variety X is the zero set of the polynomials f_i. Furthermore, since x is a nonsingular point for X, the Jacobian matrix $(\mathrm{d}f_i/\mathrm{d}z_j)_{i,j}$ has rank $n-d$. Without loss of generality, we may assume the minor $(\mathrm{d}f_i/\mathrm{d}z_j)_{1 \leq i, j \leq n-d}$ is invertible.

We let $x = (x_1, \ldots, x_n)$ be the coordinates of the point x in the above affine chart; each $x_i \in \mathfrak{o}_v$ because $x = \iota(\alpha)$. Then, $U_{\bar{\alpha}}$ is identified with points $(z_1, \ldots, z_n) \in \mathfrak{o}_v^n$ such that $z_i \equiv x_i \pmod{\pi_v}$ for π_v a generator for the maximal ideal in $\mathfrak{o}_v$. Using the Implicit

Function Theorem (see [22, Theorem 5.9, p. 19]), we see that there exists a sufficiently small p-adic neighborhood U_0 of $(x_{n-d+1}, \ldots, x_n)$, there exists a p-adic neighborhood V_0 of $(x_1, \ldots, x_{n-d})$, and there exists a p-adic analytic function $g: U_0 \longrightarrow V_0$ such that $g(x_{n-d+1}, \ldots, x_n) = (x_1, \ldots, x_{n-d})$ and, moreover, for each $\gamma \in U_0$, we have $(g(\gamma), \gamma) \in X(K_v)$. Furthermore, at the expense of shrinking both U_0 and V_0 we may assume that for each $\gamma \in U_0$, the point $(g(\gamma), \gamma)$ is in U. Since $U_0 \subset \mathbb{A}^d$ is a d-dimensional K_v-manifold we conclude that U is Zariski dense in X. ■

The following result is a consequence of Proposition 2.2 for varieties defined over number fields. Firstly, for any number field K, and any finite set S of places (containing all archimedean places), we denote by $\mathfrak{o}_{K,S}$ the subring containing all $u \in K$ such that $|u|_v \le 1$ for each $v \notin S$. Secondly, we denote by $\mathfrak{o}_K$ the ring of algebraic integers in the number field K, and for each nonarchimedean place v of K, we denote by $(\mathfrak{o}_K)_v$ the localization of $\mathfrak{o}_K$ at v. Then, for each quasiprojective variety X defined over a number field K, there exists a finite set S of places (containing all archimedean places) and there exists a $\mathfrak{o}_{K,S}$-scheme $\mathcal{X}$ whose generic fiber is isomorphic to X. In particular, we can prove the following result for $(\mathfrak{o}_K)_v$-schemes.

Proposition 2.3. Let K be a number field, let v be a nonarchimedean place of K, and let $(\mathfrak{o}_K)_v$ be the localization of $\mathfrak{o}_K$ at the place v. Let X be a quasiprojective variety defined over K, let $\mathcal{X}$ be an $(\mathfrak{o}_K)_v$-scheme whose generic fiber is isomorphic to X, let $r: \mathcal{X}((\mathfrak{o}_K)_v) \longrightarrow \bar{\mathcal{X}}(k_v)$ be the usual reduction map, and let $\iota: \mathcal{X}((\mathfrak{o}_K)_v) \longrightarrow X(K)$ be the usual map coming from base extension. Let $\alpha \in \mathcal{X}((\mathfrak{o}_K)_v)$ such that $\iota(\alpha)$ is a smooth point on X, and let U be the set of all $y \in X(\bar{K})$ such that the Zariski closure of y intersects $\bar{\mathcal{X}}$ at α_v. Then, U is Zariski dense in X. □

Proof. Let K_v be the completion of K with respect to $|\cdot|_v$, and let $\mathfrak{o}_v$ be the ring of v-adic integers of K_v. Let $\mathcal{X}_{\mathfrak{o}_v}$ be the extension of $\mathcal{X}$ to $\mathrm{Spec}(\mathfrak{o}_v)$, and let $\mathcal{X}_{K_v}$ be its generic fiber. Then, using Proposition 2.2, there exists a set $U_1 \subset \mathcal{X}_{\mathfrak{o}_v}(\mathfrak{o}_v)$ whose intersection with the generic fiber $\mathcal{X}_{K_v}$ is a Zariski dense subset of $\mathcal{X}_{K_v}$. We identify U_1 with its intersection with the generic fiber $\mathcal{X}_{K_v}$. Arguing as in the proof of Proposition 2.2, we consider a system of coordinates for an affine subset $X_1 \subset X$ containing x (also defined over K), and find an open set $U_0 \subset \mathbb{A}^d(K_v)$ and a v-adic analytic function $g: U_0 \longrightarrow K_v^{n-d}$ such that for each $z \in U_0$, we have $(g(z), z) \in X_1(K_v) \subset X(K_v)$. Furthermore, for each such point $(g(\gamma), \gamma) \in X(K_v)$, there exists a section β of $\mathcal{X}_{\mathfrak{o}_v}$ whose intersection with the special fiber is α_v, while its intersection with the generic fiber is $(g(\gamma), \gamma)$.

We let $\pi : X_1 \longrightarrow \mathbb{A}^d$ be the projection on the last d coordinates. Then, using the Fiber Dimension Theorem [33, Section 6.3], we conclude that there exists an open Zariski subset $U_2 \subseteq \mathbb{A}^d$ such that for each $\gamma \in U_2(\bar{K}) \cap U_0$, the fiber $\pi^{-1}(\gamma)$ is a $\bar{K}$-variety of dimension 0 (here, we use that X and also X_1 are defined over K). Since $U_0 \subset \mathbb{A}^d$ is a d-dimensional K_v-manifold and U_2 is the complement in $\mathbb{A}^d$ of a proper algebraic subvariety defined over $\bar{K}$, we conclude that $U_2(\bar{K}) \cap U_0$ is Zariski dense in $\mathbb{A}^d$. For each $\gamma \in U_2(\bar{K}) \cap U_0$, we have

$$(g(\gamma), \gamma) \in U_1 \cap \pi^{-1}(\gamma) \subset U_1 \cap X_1(\bar{K}).$$

Let h denote the map from $U_2(\bar{K}) \cap U_0$ to X_1 sending γ to $(g(\gamma), \gamma)$. Then, the dimension of the closure of $U_2(\bar{K}) \cap U_0$ is equal to the dimension of the closure of $h(U_2(\bar{K}) \cap U_0)$ since $\pi \circ h$ is the identity on $U_2(\bar{K}) \cap U_0$ and π is finite-to-one on $h(U_2(\bar{K}) \cap U_0)$. Since this dimension is d, which is also the dimension of X_1, we see that $h(U_2(\bar{K}) \cap U_0) \subseteq U$ is Zariski dense in X_1 and thus U is Zariski dense in X. ■

3 Burnside's Problem

In this section, we continue with the notation from Section 2 for g, p, $(K_v, |\cdot|_v)$, $\mathfrak{o}_v$, π, k_v, e and r. In addition, assume $p > 2$.

Our first result gives an upper bound for the size of torsion of the automorphism group of a quasiprojective variety X defined over a local field. So, our setup is as follows: for a $\mathfrak{o}_v$-scheme $\mathcal{X}$, we let $\bar{\mathcal{X}}$ be its special fiber (over k_v). For a point $\alpha \in \mathcal{X}(\mathfrak{o}_v)$, we let its residue class $\mathcal{U}_{\bar{\alpha}} = \{\beta \in \mathcal{X}(\mathfrak{o}_v) : \bar{\beta} = \bar{\alpha}\}$, where $\bar{\gamma} \in \bar{\mathcal{X}}(k_v)$ is the reduction modulo v of $\gamma \in \mathcal{X}(\mathfrak{o}_v)$. Finally, we note that if $\bar{\alpha}$ is a smooth point, then each $\beta \in \mathcal{U}_{\bar{\alpha}}$ is also a smooth point.

Theorem 3.1. Let $\mathcal{X}$ be a $\mathfrak{o}_v$-scheme whose generic fiber is a K-variety of dimension g, let $\mathrm{Aut}(\mathcal{X})$ be the group of $\mathfrak{o}_v$-scheme isomorphisms $\mathcal{X} \longrightarrow \mathcal{X}$, and let $G \subseteq \mathrm{Aut}(\mathcal{X})$ be a torsion group. If $\mathcal{X}(\mathfrak{o}_v)$ contains a smooth point, then G is finite and $\#G \le (\#k_v)^{g(1+e)\cdot\binom{g+e+1}{g}} \cdot \#\mathrm{GL}_g(k_v) \cdot \#\bar{\mathcal{X}}(k_v)$. □

Proof. We let $\alpha \in \mathcal{X}(k_v)$ be a smooth point and let $G_{\bar{\alpha}}$ be the subgroup of G consisting of all σ such that $\sigma\bar{\alpha} = \bar{\alpha}$. Since $[G : G_{\bar{\alpha}}] \le \#\bar{\mathcal{X}}(k_v)$, it will suffice to bound the size of $G_{\bar{\alpha}}$.

Let $\mathcal{O}_{\bar{\alpha}}$ denote the local ring of $\mathcal{X}$ at $\bar{\alpha}$, let $\mathfrak{m}_{\bar{\alpha}}$ denote its maximal ideal, let $\hat{\mathcal{O}}_{\bar{\alpha}}$ denote the completion of $\mathcal{O}_{\bar{\alpha}}$ at $\mathfrak{m}_{\bar{\alpha}}$, and let $\hat{\mathfrak{m}}_{\bar{\alpha}}$ denote the maximal ideal of $\hat{\mathcal{O}}_{\bar{\alpha}}$. Since $\alpha \in \mathcal{X}$ is smooth, the quotient $\hat{\mathcal{O}}_{\bar{\alpha}}/(\pi)$ is regular. By the Cohen structure theorem for regular local rings (see [11, Theorem 9] or [23, Theorem 29.7]), the quotient ring $\hat{\mathcal{O}}_{\bar{\alpha}}/(\pi)$ is isomorphic to a formal power series ring of the form $k_v[\![y_1,\dots,y_g]\!]$. Choosing $z_i\in\hat{\mathfrak{m}}_v$ for $i=1,\dots,g$ such that the residue class of each z_i is equal to y_i, we obtain a minimal basis $\{\pi,z_1,\dots,z_g\}$ for $\hat{\mathfrak{m}}_v$ (see [11]). Thus, we see that $\hat{\mathcal{O}}_{\bar{\alpha}}$ is naturally isomorphic to a formal power series ring $\mathfrak{o}_v[\![z_1,\dots,z_g]\!]$.

Arguing exactly as in the proof of [7, Proposition 2.2] we then obtain that there is a v-adic analytic isomorphism $\iota:\mathcal{U}_{\bar{\alpha}}\longrightarrow\mathfrak{o}_v^g$, such that for any $\sigma\in G_{\bar{\alpha}}$, there are power series $F_1,\dots,F_g\in\mathfrak{o}_v[\![z_1,\dots,z_g]\!]$ with the properties that

(i) each F_i converges on $\mathfrak{o}_v^g$;
(ii) for all $(\beta_1,\dots,\beta_g)\in\mathfrak{o}_v^g$, we have

$$\iota(\sigma(\iota^{-1}(\beta_1,\dots,\beta_g)))=(F_1(\beta_1,\dots,\beta_g),\dots,F_g(\beta_1,\dots,\beta_g));\ \text{and} \tag{3.1}$$

(iii) each F_i is congruent to a linear polynomial mod v (in other words, all the coefficients of terms of degree >1 are in the maximal ideal $\mathfrak{m}_v$ of $\mathfrak{o}_v$). Moreover, for each i, we have

$$F_i(z_1,\dots,z_g)=\frac{1}{\pi}\cdot H_i(\pi z_1,\dots,\pi z_g),$$

for some $H_i\in\mathfrak{o}_v[\![z_1,\dots,z_g]\!]$.

We write $\vec{\beta}:=(\beta_1,\dots,\beta_g)$ and $\mathcal{F}_\sigma:=\iota\sigma\iota^{-1}$, we thus have

$$\mathcal{F}_\sigma(\vec{\beta})\equiv C_\sigma+L_\sigma(\vec{\beta})\pmod{v} \tag{3.2}$$

for a $C_\sigma\in\mathfrak{o}_v^g$ and a $g\times g$ matrix L_σ with coefficients in $\mathfrak{o}_v$. Let $\bar{L}_\sigma$ be the reduction of L_σ modulo π. Since σ is an étale map of $\mathfrak{o}_v$-schemes, $\bar{L}_\sigma$ must be invertible. We define $D_{\bar{\alpha}}:G_{\bar{\alpha}}\longrightarrow\mathbb{G}_a^g(k_v)\rtimes\mathrm{GL}_g(k_v)$ by $D_{\bar{\alpha}}(\sigma)=(\overline{C_\sigma},\bar{L}_\sigma)$, where $\mathbb{G}_a^g(k_v)\rtimes\mathrm{GL}_g(k_v)$ is the group of affine transformations of k_v^g.

We clearly have $\mathcal{F}_{\sigma_1\sigma_2}=\mathcal{F}_{\sigma_1}\mathcal{F}_{\sigma_2}$ for $\sigma_1,\sigma_2\in G_{\bar{\alpha}}$. Reducing modulo π, it follows from (3.2) that $D_{\bar{\alpha}}(\sigma_1\sigma_2)=D_{\bar{\alpha}}(\sigma_1)D_{\bar{\alpha}}(\sigma_2)$. Thus, $D_{\bar{\alpha}}$ is a group homomorphism; let $G_{\bar{\alpha},1}$ be the kernel of $D_{\bar{\alpha}}$.

Next, we bound $\#G_{\bar{\alpha},1}$. We consider the map

$$E_{\bar{\alpha}}:G_{\bar{\alpha},1}\longrightarrow\mathcal{V}_g:=((\mathfrak{o}_v/\pi^{e+1}\mathfrak{o}_v)[\![z_1,\dots,z_g]\!]/(z_1,\dots,z_g)^{e+2})^g,$$

given by reducing each coordinate of $\mathcal{F}_\sigma\in G_{\bar{\alpha},1}$ modulo π^{e+1}. Using property (iii) above, we observe that $E_{\bar{\alpha}}$ is well-defined and that it satisfies $E_{\bar{\alpha}}(\sigma_1\sigma_2)=E_{\bar{\alpha}}(\sigma_1)E_{\bar{\alpha}}(\sigma_2)$.

Furthermore, because each $E_{\bar{\alpha}}(\sigma)$ for $\sigma \in G_{\bar{\alpha},1}$ is an invertible power series, we conclude that $E_{\bar{\alpha}}$ restricts to a group homomorphism from $G_{\bar{\alpha},1}$ into the subgroup of units of $\mathcal{V}_g$ (with respect to the composition of functions) consisting of $\mathcal{F}$ that satisfy the congruence $\mathcal{F}(z) \equiv z \pmod{\pi}$. Since this group of units has at most $(\#k_v)^{eg\cdot\binom{g+e+1}{g}}$ elements, we are left to show that if $\sigma \in \ker E_{\bar{\alpha}}$, then σ is the identity.

Indeed, if $\mathcal{F}_\sigma(z) \equiv z \pmod{\pi^{e+1}}$ for each $z \in \mathfrak{o}_v^g$, then $\mathcal{F}_\sigma(\vec{\beta}) \equiv \vec{\beta} \pmod{p^c}$ for some $c > 1/(p-1)$. Now fix $\vec{\beta}$; by [30, Theorem 1], there are v-adic analytic power series $\theta_1, \ldots, \theta_g \in \mathfrak{o}_v[z]$, convergent on $\mathfrak{o}_v$, such that

$$\mathcal{F}_\sigma^n(\vec{\beta}) = (\theta_1(n), \ldots, \theta_g(n))$$

for all $n \in \mathbb{N}$. Since σ has finite order, there is an N_σ such that $\mathcal{F}_\sigma^{N_\sigma}$ is the identity, we so have $\theta_i(kN_\sigma) = \beta_i$ for all $k \in \mathbb{N}$. Hence, $\theta_i(u) - \beta_i$ has infinitely many zeros $u \in \mathfrak{o}_v$. Therefore, $\theta_i(u) - \beta_i$ is identically zero since any nonzero convergent power series on $\mathfrak{o}_v$ has finitely many zeros in $\mathfrak{o}_v$. Thus, $\mathcal{F}_\sigma(\vec{\beta}) = \vec{\beta}$ for all $\vec{\beta} \in \mathfrak{o}_v^g$.

Hence, we have $\sigma(z) = z$ for all $z \in \mathcal{U}_{\bar{\alpha}}$. Since $\mathcal{U}_{\bar{\alpha}}$ is Zariski dense in $\mathcal{X}$ (they are both g-dimensional K_v-manifolds), we have that σ acts on identically on all of X. This concludes our proof. ∎

The following result is an immediate corollary of Theorem 3.1 since each torsion point of a semiabelian variety X induces a torsion element of $\mathrm{Aut}(X)$.

Corollary 3.2. Let X be a semiabelian $\mathfrak{o}_v$-scheme whose generic fiber has dimension g. Then, $\#\mathcal{X}_{\mathrm{tor}}(\mathfrak{o}_v) \le (\#k_v)^{g(1+e)\cdot\binom{g+e+1}{g}} \cdot \#\mathrm{GL}_g(k_v) \cdot \#\bar{\mathcal{X}}(k_v)$. □

If G is cyclic, then we can give a much better bound for $\#G$. In fact, Theorem 1.1 yields an upper bound for the length of the orbit of any $\mathfrak{o}_v$-subscheme $\mathcal{Y}$ of $\mathcal{X}$ which is preperiodic under the action of an étale endomorphism Φ of $\mathcal{X}$. In [18], Hutz finds upper bounds for the length of orbits of preperiodic *points* on varieties of arbitrary dimension, while Theorem 1.1 yields upper bounds for the lengths of orbits of preperiodic *subvarieties* of arbitrary dimension. We recall that r is the smallest nonnegative integer larger than $(\log(e) - \log(p-1))/\log(2)$, where e is the ramification index of $K_v/\mathbb{Q}_p$.

Proof of Theorem 1.1. We use the same setup as in the proof of Theorem 3.1. Let $\beta \in \mathcal{Y}(\mathfrak{o}_v)$ be a smooth point on $\mathcal{Y}$. Since $\bar{\mathcal{X}}(k_v)$ is finite, there is an $\ell \ge 0$ such that the residue class of $\Phi^\ell(\beta)$ is periodic under Φ; we note this residue class by U_0 and we denote $\Phi^\ell(\mathcal{Y})$ by $\mathcal{Y}'$. There is then an integer k such that $\Phi^k(U_0) = U_0$ and $k + \ell \le \#\bar{\mathcal{X}}(k_v)$.

Since $\Phi^{\ell}(\beta) \in \mathcal{Y}'(\mathfrak{o}_v) \cap U_0$ is a smooth point on the generic fiber of $\mathcal{Y}'$, Proposition 2.2 yields that $\mathcal{Y}'(\mathfrak{o}_v) \cap U_0$ is Zariski dense in $\mathcal{Y}'$. Let $x \in U_0 \cap \mathcal{Y}'(\mathfrak{o}_v)$, let $m := p^{1+r} \cdot \#\mathrm{GL}_g(k_v)$, and let $\Psi := \Phi^{mk}$. Arguing as in the proof of Theorem 3.1 (note that in order to apply the strategy from [7, Proposition 2.2], we require that Φ is étale and that x is smooth on $\mathcal{X}$ only), and also applying Proposition 2.1, we obtain that $\mathcal{F}_{\Psi}(z) \equiv z \pmod{p^c}$ for some $c > 1/(p-1)$. Hence, by [30, Theorem 1], there exists a v-adic analytic function $\mathcal{G}_{\Psi,x} : \mathfrak{o}_v \longrightarrow U_0$ such that $\mathcal{G}_{\Psi,x}(n) = \Psi^n(x)$.

Now, let F be a polynomial in the ideal of functions vanishing on $\mathcal{Y}'$. Because $\mathcal{Y}'$ is periodic, there exists a positive integer N such that $\Phi^N(\mathcal{Y}') = \mathcal{Y}'$, and thus $F(\Phi^{nN}(x)) = 0$ for each $n \in \mathbb{N}$. On the other hand, $\mathcal{G}_{\Psi,x}(n) = \Phi^{nmk}(x)$ and so, $F(\mathcal{G}_{\Psi,x}(nN)) = 0$ for all $n \in \mathbb{N}$. Since a nonzero v-adic analytic function cannot have infinitely many zeros in $\mathbb{N} \subset \mathfrak{o}_v$, we conclude that $F(\mathcal{G}_{\Psi,x}(n)) = 0$ for all $n \in \mathbb{N}$; in particular, $F(\Phi^{mk}(x)) = 0$. Thus, $\Phi^{mk}(x) \in \mathcal{Y}'$, and so $\Phi^{km}(\mathcal{Y}') = \mathcal{Y}'$. Since $k + \ell \leq \#\bar{\mathcal{X}}(k_v)$, we have that the length of the orbit of $\mathcal{Y}$ under Φ is bounded by $km + \ell \leq m \cdot \#\bar{\mathcal{X}}(k_v) = p^{1+r} \cdot \#\mathrm{GL}_g(k_v) \cdot \#\bar{\mathcal{X}}(k_v)$. ■

The following two results are simple consequences of Theorem 1.1.

Corollary 3.3. Let $\mathcal{X}$ be a $\mathfrak{o}_v$-scheme whose generic fiber X has dimension g, let $\Phi : \mathcal{X} \longrightarrow \mathcal{X}$ be étale, and let $\alpha \in \mathcal{X}(\mathfrak{o}_v)$ be a smooth preperiodic point. Then, the length of its orbit is bounded by $p^{1+r} \cdot \#\mathrm{GL}_g(k_v) \cdot \#\bar{\mathcal{X}}(k_v)$. □

Corollary 3.4. Let X be a semiabelian $\mathfrak{o}_v$-scheme whose generic fiber has dimension g. Then, each torsion point of $\mathcal{X}(\mathfrak{o}_v)$ has order bounded above by $p^{1+r} \cdot \#\mathrm{GL}_g(k_v) \cdot \#\bar{\mathcal{X}}(k_v)$. □

Our arguments above allow us to show that for any field K of characteristic 0, and for any finitely generated extension L/K, then each finitely generated torsion subgroup of $\mathrm{Aut}(L)$ fixing K is finite, that is, Burnside's problem has a positive answer. At the expense of replacing L by a finite extension and then viewing L as the function field of a geometrically irreducible quasiprojective variety defined over K, we obtain the geometric formulation of the Burnside problem from Theorem 1.2.

Proof of Theorem 1.2. Let $\sigma_1, \ldots, \sigma_m$ be a finite set of generators for H, and let K be a finitely generated field such that $X, \sigma_1, \ldots, \sigma_m$ are all defined over K. After passing to a finite extension of the base, we may assume that $X(K)$ contains a smooth point α. Let R be a finitely generated $\mathbb{Z}$-algebra containing all the coefficients of all the polynomials defining X in some projective space, along with all the coefficients of all the polynomials defining all the σ_i locally, as in the proof of [7, Theorem 4.1]. By [7, Proposition 4.3], since a finite intersection of dense open subsets is dense, we see that there is a dense open subset U of Spec$R$ such that:

(i) there is a scheme $\mathcal{X}_U$ that is quasiprojective over U, and whose generic fiber equals X;
(ii) each fiber of $\mathcal{X}_U$ is geometrically irreducible;
(iii) each σ_i extends to an automorphism σ_{iU} of $\mathcal{X}_U$;
(iv) α extends to a smooth section $U \longrightarrow \mathcal{X}_U$.

Now, arguing as in [7, Proposition 4.4], and using [6, Lemma 3.1], we see that there is an embedding of R into $\mathbb{Z}_p$ (for some prime $p \geq 5$), and a $\mathbb{Z}_p$-scheme $\mathcal{X}_{\mathbb{Z}_p}$ such that

(i) $\mathcal{X}_{\mathbb{Z}_p}$ is quasiprojective over $\mathbb{Z}_p$, and its generic fiber equals X;
(ii) both the generic and the special fiber of $\mathcal{X}_{\mathbb{Z}_p}$ are geometrically irreducible;
(iii) each σ_i extends to an automorphism $(\sigma_i)_{\mathbb{Z}_p}$ of $\mathcal{X}_{\mathbb{Z}_p}$;
(iv) α extends to a smooth section Spec$\mathbb{Z}_p \longrightarrow \mathcal{X}_{\mathbb{Z}_p}$.

Then Theorem 3.1 finishes our proof. ■

4 Bounds on the Number of Periodic Hypersurfaces

In this section, we give explicit bounds on the number of σ-periodic hypersurfaces when σ is an automorphism of an irreducible quasi-projective variety X which preserves no rational fibration. In particular, we show the number of σ-periodic hypersurfaces is finite unless there exists a nonconstant rational function f such that $f \circ \sigma = f$. Moreover, we are able to give a bound for both the lengths of periods and the number of σ-periodic hypersurfaces in terms of geometric data, although this bound depends upon the field of definition for σ. We note that Cantat [10, Theorem B] proved there exists a bound $N(\sigma)$ (depending on σ) such that if there exist more than $N(\sigma)$ irreducible periodic hypersurfaces, then σ must preserve a nonconstant rational fibration. In the case that σ is defined over a number field K and there is a point $x \in X(K)$ with a dense orbit under σ, we are able to give a bound that depends only upon the dimension of X and the Picard number of a projective closure (see Theorem 1.5). We begin with a lemma about ranks of multiplicative subgroups of a field that are stable under an automorphism of the field. As a matter of notation, for an automorphism σ of a field K, we denote by K^σ the set of all fixed points of σ.

Proposition 4.1. Let k be an algebraically closed of characteristic zero and let K be a finitely generated field extension of k. Suppose that $\sigma : K \to K$ is a k-algebra automorphism with $K^\sigma = k$. If G is a finitely generated σ-invariant subgroup of K^*, then the rank of $G/(G \cap k^*)$ is at most $\mathrm{trdeg}_k(K)$. □

Proof. Suppose, toward a contradiction, that the rank of $G/(G \cap k^*)$ is $m > \mathrm{trdeg}_k(K)$ and suppose that $x_1, \ldots, x_m$ are elements of G whose images in $G/(G \cap k^*)$ generate a free abelian group of rank m. Then, there is some nonzero polynomial $P(t_1, \ldots, t_m) \in k[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$ such that $P(x_1, \ldots, x_m) = 0$. We write P as

$$\sum_{j_1, \ldots, j_m} c_{j_1, \ldots, j_m} t_1^{j_1} \cdots t_m^{j_m}$$

and we let

$$N := \#\{(j_1, \ldots, j_m) : c_{j_1, \ldots, j_m} \neq 0\}.$$

We may take P so that $N > 1$ is minimal. By multiplying P by an appropriate monomial and nonzero constant, we may also assume that the constant coefficient of P is equal to one. Then, we have

$$P(\sigma^i(x_1), \ldots, \sigma^i(x_m)) = 0$$

for all $i \in \mathbb{Z}$. In other words, for each integer i,

$$(z_{j_1, \ldots, j_m})_{(j_1, \ldots, j_m)} = (\sigma^i(x_1^{j_1} \cdots x_m^{j_m}))_{(j_1, \ldots, j_m)} \in G^N$$

is a solution to the S-unit equation

$$\sum_{j_1, \ldots, j_m} c_{j_1, \ldots, j_m} z_{j_1, \ldots, j_m} = 0.$$

By minimality of N, each of these solutions is primitive; that is, no proper subsum vanishes. (If some proper nontrivial subsum of

$$\sum_{j_1, \ldots, j_m} c_{j_1, \ldots, j_m} \sigma^i(x_1)^{j_1} \cdots \sigma^i(x_m)^{j_m}$$

vanished for some i, then we could apply σ^{-i} to this subsum and get a smaller N, contradicting minimality.) By the theory of S-unit equations for fields of characteristic zero (see Evertse et al. [15]), we know there are only finitely many primitive solutions in G^N to the equation

$$\sum_{j_1, \ldots, j_m} c_{j_1, \ldots, j_m} z_{j_1, \ldots, j_m} = 0$$

up to multiplication by elements of G. It follows that there is some $M>0$ and some $y\in G$ such that

$$\sigma^M(x_1^{j_1}\cdots x_m^{j_m}) = yx_1^{j_1}\cdots x_m^{j_m}$$

whenever $c_{j_1,\dots,j_m}\neq 0$. Since $c_{0,\dots,0}\neq 0$, we see that $y=1$. Thus, if we pick $(j_1,\dots,j_m)\neq(0,\dots,0)$ with $c_{j_1,\dots,j_m}\neq 0$, then σ^M fixes $x_1^{j_1}\cdots x_m^{j_m}$, which by assumption is not in k^*, and so σ^M has a fixed field of transcendence degree at least one over k. Since the fixed field of σ^M is a finite extension of the fixed field of σ, we see that the fixed field of σ has transcendence degree at least one over k, a contradiction. The result follows. ■

As a corollary, we obtain the following result.

Theorem 4.2. Let K be a finitely generated extension of $\mathbb{Q}$ and let X be an irreducible quasi-projective variety defined over K. Then, there exists a positive constant $N=N(X,K)$ such that whenever $\sigma\in\mathrm{Aut}_K(X)$ has the property that there are no nonconstant $f\in\bar{K}(X)$ with $f\circ\sigma=f$ there are at most N σ-periodic hypersurfaces and they all have period at most N. Moreover, N can be taken to be $\mathrm{rank}(\mathrm{Cl}(\tilde{X}))+\dim(X)$, where $\tilde{X}$ is the normalization of X. □

We note that when Y is a normal quasi-projective variety over a finitely generated extension of $\mathbb{Q}$, we have $\mathrm{Cl}(Y)$ has finite rank [8, Lemma 5.6 (1)]. We will find it convenient to regard K as a subfield of $\mathbb{C}$ throughout.

Proof of Theorem 4.2. It is no loss of generality to assume that X is normal. Suppose that there is no nonconstant $f\in\bar{K}(X)$ with $f\circ\sigma=f$. Let $N:=\mathrm{rank}(\mathrm{Cl}(X))+\dim(X)$, and suppose that we have $N+1$ distinct σ-periodic hypersurfaces $Y_0,\dots,Y_N$. By replacing σ by an iterate, we may assume that $\sigma(Y_i)=Y_i$ for all i. By relabeling if necessary, we may assume that there is some $m\leq N-\dim(X)-1$ such that $[Y_0],\dots,[Y_m]$ generate a free $\mathbb{Z}$-module of $\mathrm{Cl}(Y)$ and that for $i>m$, $[Y_0],\dots,[Y_m],[Y_i]$ are dependent in $\mathrm{Cl}(X)$. This means that for $i\in\{N-\dim(X),\dots,N\}$, there is a principal divisor $(f_i)=c_{i,i}[Y_i]+\sum_{j=0}^m c_{i,j}[Y_j]$, where the $c_{i,j}$ are integers and $c_{i,i}$ is nonzero. By construction, we have $f_i\circ\sigma$ has the same divisor as f_i for $i=N-\dim(X),\dots,N$. Also, the f_i generate a free abelian subgroup of $\mathbb{C}(X)^*$, which can be seen by noting that the valuation on $\mathbb{C}(X)$ induced by Y_i, ν_{Y_i}, has the property that $\nu_{Y_i}(f_i)$ is nonzero but $\nu_{Y_j}(f_i)=0$ for $j\in\{N-\dim(X),\dots,N\}\setminus\{i\}$.

Since $f_i\circ\sigma$ has the same divisor as f_i, we see that $f_i\circ\sigma/f_i$ is in $\Gamma(X,\mathcal{O}_X)^*$. Let G denote the subgroup of $\mathbb{C}(X)^*/\mathbb{C}^*$ generated by $\Gamma(X,\mathcal{O}_X)^*/\mathbb{C}^*$ and by the images of the

f_i. Then, we have shown that the rank of G is at least $\dim(X)+1$. Moreover, G is finitely generated since $\Gamma(X,\mathcal{O}_X)^*/\mathbb{C}^*$ is finitely generated [8, Lemma 5.6(2)]. Furthermore, σ induces an automorphism of G since $\Gamma(X,\mathcal{O}_X)^*/\mathbb{C}^*$ is closed under application of σ and since $f_i\circ\sigma\in\Gamma(X,\mathcal{O}_X)^* f_i$. We now let $g_1,\dots,g_s$ be elements of $\mathbb{C}(X)^*$ whose images in $\mathbb{C}(X)^*/\mathbb{C}^*$ generate G. Let G_0 denote the subgroup of $\mathbb{C}(X)^*$ generated by $g_1,\dots,g_s$. Then, there exist complex numbers $\lambda_1,\dots,\lambda_s$ such that $g_i\circ\sigma\in\lambda_i G_0$. Let H denote the subgroup of $\mathbb{C}(X)^*$ generated by G_0 and by $\lambda_1,\dots,\lambda_s$. Then, H is finitely generated and by construction we have $h\circ\sigma\in H$ for all $h\in H$. Furthermore, the rank of $H/(H\cap\mathbb{C}^*)$ is at least $\dim(X)+1$, since its rank is at least as large as the rank of G. Lemma 4.1 gives a contradiction. The result follows. ■

We note that for any complex variety X with automorphism σ, there is some finitely generated extension K of $\mathbb{Q}$ such that X is defined over K and such that $\sigma\in\mathrm{Aut}_K(X)$ and so Theorem 4.2 can be applied using the value of $N(X,K)$ given in the statement of the theorem.

Also as a corollary of Theorem 4.2, we can prove that for any quasiprojective variety X defined over $\bar{\mathbb{Q}}$ under the action of an automorphism Φ which does not preserve a rational fibration, there exist nonperiodic codimension-1 subvarieties (defined over $\bar{\mathbb{Q}}$). Indeed, using Theorem 4.2, there exist finitely many codimension-1 periodic subvarieties Y_i; in addition, let $N_1\in\mathbb{N}$ such that each Y_i is fixed by Φ^{N_1}. So, we can find an algebraic point $x\in X(\bar{\mathbb{Q}})$ which is not contained in the above finitely many codimension-1 subvarieties Y_i. Then we simply take Y be the intersection of X (inside some projective space) with a hyperplane (defined over $\bar{\mathbb{Q}}$) passing through x, but not containing $\Phi^{N_1}(x)$; then Y is not periodic (since if it were, then it would be fixed by Φ^{N_1} but on the other hand, $\Phi^{N_1}(x)\notin Y(\bar{\mathbb{Q}})$), and therefore its orbit under Φ is Zariski dense in X.

5 Subvarieties with Zariski Dense Orbits

The setup for this Section is as follows: X is a quasiprojective variety defined over $\mathbb{C}$, and Φ is an automorphism of X that preserves no nonconstant rational fibration. Our goal is to prove Theorem 1.3; we use Theorems 1.1 and 4.2.

Proof of Theorem 1.3**.** Arguing as before, for a suitable prime $p\geq 5$, we find a $\mathbb{Z}_p$-scheme $\mathcal{X}$ such that

(i) X is the generic fiber of $\mathcal{X}$, while the special fiber $\bar{\mathcal{X}}$ of $\mathcal{X}$ is a geometrically irreducible quasiprojective variety;

(ii) Φ extends to an automorphism of $\mathcal{X}$;

(iii) there exists $x_0 \in \mathcal{X}(\mathbb{Z}_p)$ such that its reduction $\overline{x_0}$ modulo p is a smooth point of $\bar{\mathcal{X}}$.

Let $U_0 := \{x \in \mathcal{X}(\mathbb{Z}_p) : \bar{x} = \overline{x_0}\}$ be the residue class of x_0 (since $\overline{x_0}$ is a smooth point on $\bar{\mathcal{X}}$, then each $x \in U_0$ is also smooth on $\mathcal{X}$). Furthermore, we identify each section in U_0 with its intersection with the generic fiber X. Using Theorem 1.1, there exists a positive integer N_1 such that each periodic subvariety Y which contains a point from U_0 which is smooth also on Y has period bounded above by N_1.

By Theorem 4.2, there exist at most finitely many codimension-1 subvarieties which are fixed by Φ^{N_1}. Let Y_1 be the union of all these codimension-1 subvarieties. On the other hand, by the definition of N_1, if $x \in U_0$ is (pre)periodic, then $\Phi^{N_1}(x) = x$. Because Φ has infinite order (since it preserves no nonconstant rational fibration), the vanishing locus for the equation $\Phi^{N_1}(x) = x$ is a proper subvariety Y_0 of $\mathcal{X}$. In conclusion, $Y_0 \cup Y_1$ is a proper subvariety of X and therefore, there exists a Zariski dense set of points $x \in U_0 \setminus (Y \cup Y_1)(\mathbb{Z}_p)$ (because U_0 is a p-adic manifold of dimension larger than $\dim(Y_0 \cup Y_1)$). Furthermore, we can choose $x \in X(\bar{\mathbb{Q}})$ by Proposition 2.3; finally, note that x is smooth since it is in U_0.

For each such point $x \in X(\bar{\mathbb{Q}}) \cap U_0$ which is not contained in $Y_0 \cup Y_1$, we can find a codimension-2 subvariety Y (defined over $\bar{\mathbb{Q}}$) whose orbit under Φ is Zariski dense in X. Indeed, we consider X embedded into a large projective space $\mathbb{P}^m$ and then intersect X with two (generic) hyperplane sections H_1 and H_2 (defined over $\bar{\mathbb{Q}}$) which pass through x, but not through $\Phi^{N_1}(x)$ (note that $\Phi^{N_1}(x) \neq x$ because $x \notin Y_0$). Furthermore, since H_1 and H_2 are generic sections passing through x, then $Y := X \cap H_1 \cap H_2$ is a codimension-2 irreducible subvariety defined over $\bar{\mathbb{Q}}$, and moreover $x \in Y$ is a smooth point. We claim that Y is not periodic under Φ. Otherwise, since Y intersects U_0, then it must be fixed by Φ^{N_1} (by Theorem 1.1 and our choice for N_1). However, $x \in Y$, but $\Phi^{N_1}(x) \notin Y$, which shows that Y is not fixed by Φ^{N_1}, and thus Y is not periodic under the action of Φ. Let Z be the Zariski closure of the orbit of Y under the action of Φ. Since Y is not periodic under Φ, then $\dim(Z) > \dim(Y)$. Now, if $\dim(Z) < \dim(X)$, then Z is a codimension-1 subvariety, and in addition it is fixed by Φ^{N_1}. Then it has to be contained in Y_1. However, this is impossible since $x \in Z$ but $x \notin Y_1$. In conclusion, $Z = X$, as desired. ■

In particular, if the codimension-2 subvariety Y from the conclusion of Theorem 1.3 has the property that $Y(L)$ is Zariski dense in Y (for some number field L containing the field of definition for Φ), then $X(L)$ is Zariski dense in X. So, our

Theorem 1.3 may be used to prove that certain varieties X have a Zariski dense set of rational points, by reducing the problem to finding a potentially dense set of rational points on a codimension-2 subvariety Y of X.

In the case that $\sigma : X \to X$ is defined over a number field K and there is a point $x \in X(K)$ with dense orbit under σ, we obtain a much stronger upper bound (that has no dependence on the number field) for the period of codimension-1 subvarieties of X periodic under the automorphism.

Proof of Theorem 1.5. We extend σ to a map $\mathcal{X}' \longrightarrow \mathcal{X}'$, where $\mathcal{X}'$ is defined over the ring of integers $\mathfrak{o}_K$. Let R be the localization of $\mathfrak{o}_K$ away from all at the primes of bad reduction. Then, we obtain an automorphism of R-schemes $\sigma_0 : \mathcal{X} \longrightarrow \mathcal{X}$. Now, let $\mathcal{Y}$ be some projective closure for $\mathcal{X}$; then x meets $\mathcal{Y} \setminus \mathcal{X}$ over at most finitely many finite primes, call this set T, and let R' denote the localization of R away from T. Let Y be the generic fiber of $\mathcal{Y}$.

Let W be an invariant subvariety of X. Suppose that W has at least $\dim X - h^1(Y, \mathcal{O}_Y) + \rho + 1$ geometric components, where ρ is the Picard number of Y (the rank of its Néron–Severi group). Then, clearly x is not in W so there is an at most finite set T' of primes at which x meets W. Let $S = T \cup T' \cup (\operatorname{Spec} \mathfrak{o}_K \setminus \operatorname{Spec} R)$. Then, x is S-integral relative to W and, since $\sigma^{-1}(W) = W$, we see that $\sigma^n(x)$ is S-integral relative to W for all n (if $\sigma^n(x)$ met W modulo a prime, then x would meet $\sigma^{-n}(W)$ modulo that same prime). But by a result of Vojta [34, Corollary 0.3], this would mean that the orbit of x was not dense, since W has at least $\dim X - h^1(Y, \mathcal{O}_Y) + \rho + 1$ geometric components, which gives a contradiction. ■

6 Other Questions

Poonen [29] has proposed a variant of Morton–Silverman's uniform boundedness conjecture, where the morphisms vary across a general families of self-maps of varieties rather than just the universal family of degree-d self-maps $\mathbb{P}^N \longrightarrow \mathbb{P}^N$. In Poonen's set-up, some fibers may have infinitely many preperiodic points. Although that cannot happen in the case of preperiodic points of morphisms $\mathbb{P}^n \longrightarrow \mathbb{P}^n$ (because of Northcott's theorem), a morphism $\mathbb{P}^n \longrightarrow \mathbb{P}^n$ *can* have infinitely many positive-dimensional periodic subvarieties. For example, if f is a homogeneous two-variable polynomial of degree n, then the morphism $\mathbb{P}^2 \longrightarrow \mathbb{P}^2$ given by $[x : y : z] \mapsto [f(x, z) : f(y, z) : z^n]$ has infinitely many f-invariant curves of the form $[xz^{n^k-1} : f^k(x, z) : z^{n^k}]$, where f^k is the homogenized kth iterate of the dehomogenized one-variable polynomial $x \mapsto f(x, 1)$. On the other hand, it is possible that one may be able to bound the periods of the f-periodic subvarieties in general.

To state our question, we will need a little terminology. To be clear, we will say that V is a K-subvariety of X if V is a geometrically irreducible subvariety of X defined over K. Since so little is known about this question, we will ask it in slightly less generality than Poonen uses. Given a morphism $\Phi : X \longrightarrow X$ and a K-subvariety V of X such that V is periodic under the action of Φ, we define $\mathrm{Per}_\Phi(V)$ to be the smallest n such that $\Phi^n(V) \subseteq V$.

Question 6.1. Let $\pi : \mathcal{F} \longrightarrow S$ be a morphism of varieties defined over a number field K and let $\Phi : \mathcal{F} \longrightarrow \mathcal{F}$ be an S-morphism. For $s \in S(K)$, we let $\mathcal{F}_s$ be the fiber $\phi^{-1}(s)$ and let Φ_s be the restriction of Φ to $\mathcal{F}_s$. Is there a constant $N_{\mathcal{F}}$ such that for any $s \in S(K)$ and any periodic K-subvariety V of $\mathcal{F}_s$, we have $\mathrm{Per}_{\Phi_s}(V) \leq N_{\mathcal{F}}$? □

Even in the case where one can assign canonical heights to subvarieties of X, there may be subvarieties of X of positive dimension having canonical height 0 that are not preperiodic (see [16]). Thus, we do not know the answer to Question 6.1 even in the case of a constant family of maps.

Question 6.2. Let $\Phi : X \longrightarrow X$ be a morphism of varieties defined over a number field K. Is there a constant N_X such that for any periodic K-subvariety V of X, we have $\mathrm{Per}_\Phi(V) \leq N_X$? □

We may also ask an analog of Question 6.1 for finite subgroups of automorphism groups.

Question 6.3. Let $\pi : \mathcal{F} \longrightarrow S$ be a morphism of varieties defined over a number field K. For $s \in S(K)$, we let $\mathcal{F}_s$ denote the fiber $\pi^{-1}(s)$. Must the set

$$\{n \mid \text{ there is an } s \in S(K) \text{ such that } \mathrm{Aut}(\mathcal{F}_s) \text{ has a subgroup of order } n\}$$

be finite? □

The theorems of Mazur [24] and Merel [26] show that Questions 6.1 and 6.3 have a positive answer when $\mathcal{X}$ is a family of elliptic curves. Similarly, work of Kondō [20] shows that Question 6.3 has a positive answer when $\mathcal{F}$ is a family of K3 surfaces.

As with Question 6.1, we do not know the answer to Question 6.3 even in the constant family case. On the other hand, the bound in Theorem 3.1 depends only on the dimension of X and the number of points in the special fiber of X at the place v; by the Weil bounds of Deligne [12, 13], the number of points on this special fiber can be bounded in terms of $\#k_v$, the dimension of X, and the Betti numbers of X. Thus, one might expect that there is a bound on the largest finite subgroup of $\mathrm{Aut}(\mathcal{F}_s)$ having good reduction at v as $\mathcal{F}_s$ varies in a family.

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