THE DYNAMICAL MORDELL-LANG CONJECTURE

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Abstract. We prove a special case of a dynamical analogue of the classical Mordell-Lang conjecture. In particular, let ϕ be a rational function with no superattracting periodic points other than exceptional points. If the coefficients of ϕ are algebraic, we show that the orbit of a point outside the union of proper preperiodic subvarieties of $(\mathbb{P}^1)^g$ has only finite intersection with any curve contained in $(\mathbb{P}^1)^g$. Our proof uses results from p-adic dynamics together with an integrality argument.

1. Introduction

Let X be a variety over the complex numbers $\mathbb{C}$, let $\Phi : X \longrightarrow X$ be a morphism, and let V be a subvariety of X. For any integer $m \geq 0$, denote by Φ^m the m^{th} iterate $\Phi \circ \cdots \circ \Phi$. If $\alpha \in X(\mathbb{C})$ has the property that there is some integer $\ell \geq 0$ such that $\Phi^\ell(\alpha) \in W(\mathbb{C})$, where W is a periodic subvariety of V, then there are infinitely many integers $n \geq 0$ such that $\Phi^n(\alpha) \in V$. More precisely, if $k \geq 1$ is the period of W (the smallest positive integer m for which $\Phi^m(W) = W$), then $\Phi^{nk+\ell}(\alpha) \in W(\mathbb{C}) \subseteq V(\mathbb{C})$ for all integers $n \geq 0$. It is natural then to pose the following question: given $\alpha \in X(\mathbb{C})$, if there are infinitely many integers $m \geq 0$ such that $\Phi^m(\alpha) \in V(\mathbb{C})$, are there necessarily integers $k \geq 1$ and $\ell \geq 0$ such that $\Phi^{nk+\ell}(\alpha) \in V(\mathbb{C})$ for all integers $n \geq 0$?

This question has a positive answer in many special cases. When X is a semiabelian variety and Φ is a multiplication-by-m map, this follows from Faltings' [Fal94] and Vojta's proof [Voj96] of the Mordell-Lang conjecture in characteristic 0. More generally, the question has a positive answer when Φ is any endomorphism of a semiabelian variety (see [GTa]). Denis [Den94] treated the general question under the additional hypothesis that the integers n for which $\Phi^n(\alpha) \in V(\mathbb{C})$ are sufficiently dense in the set of all positive integers; he also obtained results for automorphisms of projective space without using this additional hypothesis. Bell [Bel06] later solved the problem completely in the case of automorphisms of affine space. In [GTb], a general framework for attacking the problem is developed and the following conjecture is made.

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Conjecture 1.1. *Let $f_1, \dots, f_g \in \mathbb{C}[t]$ be polynomials, let Φ be their action coordinatewise on $\mathbb{A}^g$, let $\mathcal{O}_\Phi((x_1, \dots, x_g))$ denote the Φ-orbit of the point $(x_1, \dots, x_g) \in \mathbb{A}^g(\mathbb{C})$, and let V be a subvariety of $\mathbb{A}^g$. Then V intersects $\mathcal{O}_\Phi((x_1, \dots, x_g))$ in at most a finite union of orbits of the form $\mathcal{O}_{\Phi^k}(\Phi^\ell(x_1, \dots, x_g))$, for some nonnegative integers k and ℓ.*

See Section 2 for the definition of the orbit $\mathcal{O}_\Phi(\alpha)$. Note that the orbits for which $k = 0$ are singletons, so that the conjecture allows not only infinite forward orbits but also finitely many extra points.

Note also that if Conjecture 1.1 holds for a given map Φ, variety V, and non-preperiodic point $\alpha = (x_1, \dots, x_g)$, and if V intersects the Φ-orbit of α in infinitely many points, then V must contain a positive-dimensional subvariety V_0 that is periodic under Φ. Indeed, the conjecture says that there are integers $k \geq 1$ and $\ell \geq 0$ such that $\Phi^{nk+\ell}(\alpha)$ lies on V for all $n \geq 0$. Since α is not preperiodic, the set $S = \{\Phi^{nk+\ell}(\alpha)\}_{n\geq 0}$ is infinite, and therefore its Zariski closure V_0' contains positive-dimensional components. Thus, if we let V_0 be the union of the positive-dimensional irreducible subvarieties of V_0', then V_0 is positive-dimensional and fixed by Φ^k, as claimed.

Conjecture 1.1 has been proved in the case that $g = 2$ and V is a line in $\mathbb{A}^2$ (see [GTZ]). The technique used there does not, however, appear to work for more general subvarieties of affine space.

Conjecture 1.1 fits into Zhang's far-reaching system of dynamical conjectures [Zha06]. Zhang's conjectures include dynamical analogues of the Manin-Mumford and Bogomolov conjectures for abelian varieties (now theorems of Raynaud [Ray83a, Ray83b], Ullmo [Ull98], and Zhang [Zha98]), as well as a conjecture about the Zariski density of orbits of points under fairly general maps from a projective variety to itself. This latter conjecture of Zhang takes the following form in the case of polynomial actions on $\mathbb{A}^g$.

Conjecture 1.2. *Let $f_1, \dots, f_g \in \overline{\mathbb{Q}}[t]$ be polynomials of the same degree $d \geq 2$, and let Φ be their action coordinatewise on $\mathbb{A}^g$. Then there is a point $(x_1, \dots, x_g) \in \mathbb{A}^g(\overline{\mathbb{Q}})$ such that $\mathcal{O}_\Phi((x_1, \dots, x_g))$ is Zariski dense in $\mathbb{A}^g$.*

Conjectures 1.2 and 1.1 may be thought of as complementary. Conjecture 1.2 posits that there is a point in $\mathbb{A}^g$ outside the union of the preperiodic proper subvarieties of $\mathbb{A}^g$ under the action of Φ, while Conjecture 1.1 asserts if a point α lies outside this union of preperiodic subvarieties, then the orbit of α under Φ intersects any subvariety V of $\mathbb{A}^g$ in at most finitely many points. We view our Conjecture 1.1 as an analogue of the classical Mordell-Lang conjecture for arithmetic dynamics where groups of rank one are replaced by single orbits.

In this paper, we prove Conjecture 1.1 over number fields for curves embedded in $\mathbb{A}^g$ under the diagonal action of any polynomial which has no periodic superattracting points. (For a definition of superattracting points, see Section 2.) In fact, we prove the following more general statement.

Theorem 1.3. *Let $C \subset (\mathbb{P}^1)^g$ be a curve defined over $\overline{\mathbb{Q}}$, and let $\Phi := (\varphi, \ldots, \varphi)$ act on $(\mathbb{P}^1)^g$ coordinatewise, where $\varphi \in \overline{\mathbb{Q}}(t)$ is a rational function with no periodic superattracting points other than exceptional points. Let $\mathcal{O}$ be the Φ-orbit of a point $(x_1, \ldots, x_g) \in (\mathbb{P}^1)^g(\overline{\mathbb{Q}})$. Then $C(\overline{\mathbb{Q}}) \cap \mathcal{O}$ is a union of at most finitely many orbits of the form $\{\Phi^{nk+\ell}(x_1, \ldots, x_g)\}_{n \geq 0}$ for nonnegative integers k and ℓ.*

See Section 2 for a definition of exceptional points.

When the function φ is a quadratic polynomial, we can prove a similar result for subvarieties of any dimension.

Theorem 1.4. *Let $V \subset (\mathbb{P}^1)^g$ be a subvariety defined over $\overline{\mathbb{Q}}$, and let $\Phi := (f, \ldots, f)$ act on $(\mathbb{P}^1)^g$ coordinatewise, where $f \in \overline{\mathbb{Q}}[t]$ is a quadratic polynomial with no periodic superattracting points in $\overline{\mathbb{Q}}$. Let $\mathcal{O}$ be the Φ-orbit of a point $(x_1, \ldots, x_g) \in (\mathbb{P}^1)^g(\overline{\mathbb{Q}})$. Then $V(\overline{\mathbb{Q}}) \cap \mathcal{O}$ is a union of at most finitely many orbits of the form $\{\Phi^{nk+\ell}(x_1, \ldots, x_g)\}_{n \geq 0}$ for nonnegative integers k and ℓ.*

For quadratic polynomials over the rational numbers, we can remove the hypothesis on superattracting points and obtain a stronger result.

Theorem 1.5. *Let $V \subset (\mathbb{P}^1)^g$ be a subvariety defined over $\mathbb{Q}$, and let $\Phi := (f, \ldots, f)$ act on $(\mathbb{P}^1)^g$ coordinatewise, where $f \in \mathbb{Q}[t]$ is a quadratic polynomial. Let $\mathcal{O}$ be the Φ-orbit of a point $(x_1, \ldots, x_g) \in (\mathbb{P}^1)^g(\mathbb{Q})$. Then $V(\mathbb{Q}) \cap \mathcal{O}$ is a union of at most finitely many orbits of the form $\{\Phi^{nk+\ell}(x_1, \ldots, x_g)\}_{n \geq 0}$ for nonnegative integers k and ℓ.*

The strategy used for proving the theorems above is based in part on a technique first used by Skolem [Sko34] (and later extended by Mahler [Mah35] and Lech [Lec53]) to treat linear recurrence sequences. The idea is to find an infinite arithmetic sequence $\mathcal{S}$ of integers such that there are infinitely many $m \in \mathcal{S}$ with $\Phi^m(\alpha)$ lying on V, and then to construct a nonarchimedean p-adic analytic map θ sending $\mathcal{S}$ into $\mathbb{A}^g(\mathbb{C}_p)$ such that $\theta(m) = \Phi^m(x_1, \ldots, x_g)$ for each integer m in the sequence. Then, for any polynomial F that vanishes on V, we have $F(\theta(k)) = 0$ for infinitely many k. Since the zeros of a nonzero p-adic analytic function are isolated, $F \circ \theta$ must vanish at *all* k in the sequence. Work of Rivera-Letelier [RL03] from p-adic dynamics shows that there is such an arithmetic sequence $\mathcal{S}$ and p-adic analytic function θ whenever there is a positive integer ℓ such that $\varphi^\ell(x_i)$ is in a p-adic quasiperiodicity disk for each i. (For a definition of quasiperiodicity disk, see Section 3.) One cannot expect every place to admit such an integer ℓ, but in many cases, diophantine techniques can be used to show that at least one such place exists.

We note that the Skolem-Mahler-Lech technique has played a role in other work done on this subject. Bell's [Bel06] and Denis's [Den94] work on automorphisms may be viewed as algebro-geometric realizations of the Skolem-Mahler-Lech theorem. Moreover, the Skolem-Mahler-Lech theorem is used in the proof of [GTa]. Evertse, Schlickewei, and Schmidt [ESS02] have given a strong quantitative version of the Skolem-Mahler-Lech theorem. It may be possible to use their result to give more precise versions of the theorems of this paper.

The proof of Theorem 1.3 shows a surprising interplay between arithmetic geometry and p-adic dynamics. In particular, we use intersection theory on $\mathbb{P}^1 \times \mathbb{P}^1$ together with an application of the classical Siegel's theorem (see Section 4) combined with the classification of the disks contained in the Fatou set of the dynamical system associated to a rational function on $\mathbb{P}^1(\mathbb{C}_p)$ (see Section 3). We note that our methods are conceptually different than the methods used in [GTZ] (which involved Ritt's classification for functional decomposition of complex polynomials).

We exclude the case that the rational function φ has superattracting points because we have been unable, thus far, to extend the method of Skolem-Mahler-Lech to this situation. There is a logarithm associated to a φ (see [GTb]) in a neighborhood of a superattracting point but this map fails to "linearize" φ in the desired manner. This linearization is exactly what Rivera-Letelier's work [RL03] provides us with in quasiperiodicity disks (see Section 3). While superattracting points are, in fact, quite simple from a dynamical perspective, it is perhaps not surprising that they present difficulties in this more diophantine context. In the cases of endomorphisms of semiabelian varieties (see [Voj96, Fal94, GTa]) and of automorphisms of affine space (see [Den94, Bel06]), the underlying maps are étale and hence have no ramification. When φ has no superattracting points, the ramification indices of φ^n remain bounded for all n; this fact plays an important role in Section 4. However, when φ has a superattracting point, these indices may become arbitrarily large. Hence, the ramification of the iterates of φ is more complicated when φ has a superattracting point.

In the polynomial case, it should be possible to extend Theorem 1.3 to the complex numbers, using Scanlon's conjectural description of periodic curves of $\mathbb{A}^2$ [Sca07], which comes from Medvedev's classification [Med07] of group-like σ-degree 1 minimal sets in the model theory of ACFA. We intend to return to this problem in a future paper.

More generally, we believe that there should be a broader *Mordell-Lang principle* which holds for any sufficiently *rigid* space X (i.e. the space does not have a large set of endomorphisms). This principle would say that any definable subset of X (in the sense of model theory; for algebraic geometry, the definable sets are algebraic varieties) intersects the orbit of a point $P \in X$ under an endomorphism Φ of X in at most finitely many orbits of the form $\{\Phi^{nk+\ell}(P)\}_{n\geq 0}$, for some nonnegative integers k and ℓ. If X is a semiabelian variety, the above principle can be found at the heart of the classical Mordell-Lang conjecture (see [GTa]). If X is $\mathbb{A}^g$ under the action of polynomial maps f_i on each coordinate, then we recover our Conjecture 1.1. Note that in either case, X has few endomorphisms. If X is semiabelian, then $\mathrm{End}(X)$ is a finitely generated integral extension of $\mathbb{Z}$. Similarly, if X is $\mathbb{A}^g$

under a polynomial action, then $(H_1, \ldots, H_g)$ is an endomorphism if and only if $H_i \circ f_i = f_i \circ H_i$ for each i, which typically implies that H_i and f_i have a common iterate. (See the extensive work on this subject by Fatou [Fat21, Fat23], Julia [Jul22], Eremenko [Ere90], among many others).

The outline of our paper is as follows. In Section 2 we introduce our notation. Sections 3 and 4 provide necessary lemmas from p-adic dynamics and from intersection theory on arithmetic surfaces. In Section 5, we prove Theorem 1.3. In Section 6, we use the results of Section 5 to prove Theorems 1.4 and 1.5.

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2. Notation

We write $\mathbb{N}$ for the set of nonnegative integers. If K is a field, we write $\overline{K}$ for an algebraic closure of K. Given a prime number p, the field $\mathbb{C}_p$ will denote the completion of an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, the field of p-adic rationals. We denote by $|\cdot|:=|\cdot|_p$ the usual absolute value on $\mathbb{C}_p$. Given $a \in \mathbb{C}_p$ and $r > 0$, we write $D(a,r)$ and $\overline{D}(a,r)$ for the open disk and closed disk (respectively) of radius r centered at a.

If K is a number field, we let $\mathfrak{o}_K$ be its ring of algebraic integers, and we fix an isomorphism π between $\mathbb{P}^1_K$ and the generic fibre of $\mathbb{P}^1_{\mathfrak{o}_K}$. For each nonarchimedean place v of K, we let k_v be the residue field of K at v, and for each $x \in \mathbb{P}^1(K)$, we let $x_v := r_v(x)$ be the intersection of the Zariski closure of $\pi(x)$ with the fibre above v of $\mathbb{P}^1_{\mathfrak{o}_K}$. (Intuitively, x_v is x modulo v.) This map $r_v : \mathbb{P}^1(K) \longrightarrow \mathbb{P}^1(k_v)$ is the *reduction map* at v.

If $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ is a morphism defined over the field K, then (fixing a choice of homogeneous coordinates) there are relatively prime homogeneous polynomials $F, G \in K[X,Y]$ of the same degree $d = \deg \varphi$ such that $\varphi([X,Y]) = [F(X,Y) : G(X,Y)]$. (In affine coordinates, $\varphi(t) = F(t,1)/G(t,1) \in K(t)$ is a rational function in one variable.) Note that by our choice of coordinates, F and G are uniquely defined up to a nonzero constant multiple. We will need the notion of good reduction of φ, first introduced by Morton and Silverman in [MS94].

Definition 2.1. *Let K be a field, let v be a nonarchimedean valuation on K, let $\mathfrak{o}_v$ be the ring of v-adic integers of K, and let k_v be the residue field at v. Let $\varphi : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ be a morphism over K, given by $\varphi([X,Y]) = [F(X,Y) : G(X,Y)]$, where $F, G \in \mathfrak{o}_v[X,Y]$ are relatively prime homogeneous polynomials of the same degree such that at least one coefficient of F or G is a unit in $\mathfrak{o}_v$. Let $\varphi_v := [F_v, G_v]$, where $F_v, G_v \in k_v[X,Y]$ are the reductions of F and G modulo v. We say that φ has* good reduction *at v if $\varphi_v : \mathbb{P}^1(k_v) \longrightarrow \mathbb{P}^1(k_v)$ is a morphism of the same degree as φ.*

If $\varphi \in K[t]$ is a polynomial, we can give the following elementary criterion for good reduction: φ has good reduction at v if and only if all coefficients of φ are v-adic integers, and its leading coefficient is a v-adic unit.

Definition 2.2. *Two rational functions φ and ψ are* conjugate *if there is a linear fractional transformation μ such that $\varphi = \mu^{-1} \circ \psi \circ \mu$.*

In the above definition, if φ and ψ are polynomials, then we may assume that μ is a polynomial of degree one.

Definition 2.3. *If K is a field, and $\varphi \in K(t)$ is a rational function, then $z \in \mathbb{P}^1(\overline{K})$ is a* periodic point *for φ if there exists an integer $n \geq 1$ such that $\varphi^n(z) = z$. The smallest such integer n is the* period *of z, and $\lambda = (\varphi^n)'(z)$ is the* multiplier *of z. If $\lambda = 0$, then z is called* superattracting. *If $|\cdot|_v$ is an absolute value on K, and if $|\lambda|_v < 1$, then z is called* attracting.

If z is a periodic point of $\varphi = \mu^{-1} \circ \psi \circ \mu$, then $\mu(z)$ is a periodic point of ψ with the same multiplier. In particular, we can define the multiplier of a periodic point at $z = \infty$ by changing coordinates.

Whether or not z is periodic, we say z is a *ramification point* or *critical point* of φ if $\varphi'(z) = 0$. If $\varphi = \mu^{-1} \circ \psi \circ \mu$, then z is a critical point of φ if and only if $\mu(z)$ is a critical point of ψ; in particular, coordinate change can again be used to determine whether $z = \infty$ is a critical point. Note that a periodic point z is superattracting if and only if at least one of $z, \phi(z), \phi^2(z), \ldots, \phi^{n-1}(z)$ is critical, where n is the period of z.

Let $\varphi : V \longrightarrow V$ be a map from a variety to itself, and let $z \in V(\overline{K})$. The (*forward*) *orbit* $\mathcal{O}_\varphi(z)$ of z under φ is the set $\{\varphi^k(z) : k \in \mathbb{N}\}$. We say z is *preperiodic* if $\mathcal{O}_\varphi(z)$ is finite. If μ is an automorphism of V, and if $\varphi = \mu^{-1} \circ \psi \circ \mu$, note that $\mathcal{O}_\varphi(z) = \mu^{-1}(\mathcal{O}_\psi(\mu(z)))$.

We say z is *exceptional* (or *totally invariant*) if there are only finitely many points w such that $z \in \mathcal{O}_\varphi(w)$ (i.e. the backward orbit of z contains only finitely many points). It is a classical result in dynamics that a morphism $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ of degree larger than one has at most two exceptional points. Moreover, it has exactly two if and only if φ is conjugate to the map $t \mapsto t^n$, for some integer $n \in \mathbb{Z}$; and it has exactly one if and only if φ is conjugate to a polynomial but not to any map $t \mapsto t^n$. In particular, φ has at least one exceptional point if and only if φ^2 is conjugate to a polynomial.

3. Quasiperiodicity disks in p-adic dynamics

As in [GTb], we will need a result on non-preperiodic points over local fields. By an *open disk in* $\mathbb{P}^1(\mathbb{C}_p)$, we will mean either an open disk in $\mathbb{C}_p$ or the complement (in $\mathbb{P}^1(\mathbb{C}_p)$) of a closed disk in $\mathbb{C}_p$. Equivalently, an open disk in $\mathbb{P}^1(\mathbb{C}_p)$ is the image of an open disk $D(0, r) \subseteq \mathbb{C}_p$ under a linear fractional transformation $\gamma \in \mathrm{PGL}(2, \mathbb{C}_p)$. Closed disks are defined similarly.

The following definition is borrowed from [RL03, Section 3.2], although we have used a simpler version that suffices for our purposes.

Definition 3.1. *Let p be a prime, let $r > 0$, let $\gamma \in \mathrm{PGL}(2, \mathbb{C}_p)$, and let $U = \gamma(D(0,r))$. Let $f : U \to U$ be a function such that*

$$\gamma^{-1} \circ f \circ \gamma(t) = \sum_{i \geq 0} c_i t^i \in \mathbb{C}_p[[t]],$$

with $|c_0| < r$, $|c_1| = 1$, and $|c_i| r^i \leq r$ for all $i \geq 1$. Then we say U is a quasiperiodicity disk *for f.*

The conditions on f in Definition 3.1 mean precisely that f is rigid analytic and maps U bijectively onto U. In particular, the preperiodic points of f in U are in fact periodic. By [RL03, Corollaire 3.12], our definition implies that U is indeed a quasiperiodicity domain of f in the sense of [RL03, Définition 3.7].

The main result of this section is the following.

Theorem 3.2. *Let p be a prime and $g \geq 1$. For each $i = 1, \dots, g$, let U_i be an open disk in $\mathbb{P}^1(\mathbb{C}_p)$, and let $f_i : U_i \to U_i$ be a map for which U_i is a quasiperiodicity disk. Let Φ denote the action of $f_1 \times \cdots \times f_g$ on $U_1 \times \cdots \times U_g$, let $\alpha = (x_1, \dots, x_g) \in U_1 \times \cdots \times U_g$ be a point, and let $\mathcal{O}$ be the Φ-orbit of α. Let V be a subvariety of $(\mathbb{P}^1)^g$ defined over $\mathbb{C}_p$. Then $V(\mathbb{C}_p) \cap \mathcal{O}$ is a union of at most finitely many orbits of the form $\{\Phi^{nk+\ell}(\alpha)\}_{n \geq 0}$ for nonnegative integers k and ℓ.*

The proof of Theorem 3.2 relies on the following lemma from p-adic dynamics, which in turn follows from the theory of quasiperiodicity domains in [RL03, Section 3.2].

Lemma 3.3. *Let $U \subseteq \mathbb{C}_p$ be an open disk, let $f : U \to U$ be a map for which U is a quasiperiodicity disk, and let $x \in U$ be a non-periodic point. Then there exist an integer $k \geq 1$, radii $r > 0$ and $s \geq |k|_p$, and, for every integer $\ell \geq 0$, a bijective rigid analytic function $h_\ell : \overline{D}(0,s) \to \overline{D}(f^\ell(x), r)$, with the following properties:*

(i) *$h_\ell(0) = f^\ell(x)$, and*
(ii) *for all $z \in \overline{D}(f^\ell(x), r)$ and $n \geq 0$, we have*

$$f^{nk}(z) = h_\ell(nk + h_\ell^{-1}(z)).$$

Proof. Write $U = D(a, R)$. By [RL03, Proposition 3.16(2)], there is an integer $k \geq 1$ and a neighborhood $U_x \subseteq U$ of x on which f^k is (analytically and bijectively) conjugate to $t \mapsto t + k$. That is, there are radii $r, s > 0$ (with $r < R$ and $s \geq |k|_p$) and a bijective analytic function $h_0 : \overline{D}(0,s) \to \overline{D}(x,r)$ such that $f^{nk}(z) = h_0(nk + h_0^{-1}(z))$ for all $z \in \overline{D}(x,r)$ and $n \geq 0$.

For each nonnegative integer ℓ, note that f^ℓ is a bijective analytic function from $\overline{D}(x,r)$ onto $\overline{D}(f^\ell(x), r)$. Thus, if we let $h_\ell := f^\ell \circ h_0$, then h_ℓ is a bijective analytic function from $\overline{D}(0,s)$ onto $\overline{D}(f^\ell(x), r)$. Moreover, for all $z \in \overline{D}(f^\ell(x), r)$, if we let $\zeta = f^{-\ell}(z) \in \overline{D}(x,r)$, then for every $n \geq 0$,

$$f^{nk}(z) = f^\ell(f^{nk}(\zeta)) = f^\ell(h_0(nk + h_0^{-1}(\zeta))) = h_\ell(nk + h_\ell^{-1}(z)).$$

Finally, replacing $h_\ell(z)$ by $h_\ell(z + h_\ell^{-1}(f^\ell(x)))$, we can also ensure that $h_\ell(0) = f^\ell(x)$. □

We are now ready to prove Theorem 3.2.

Proof of Theorem 3.2. By applying linear fractional transformations γ_i to each U_i, we may assume without loss of generality that each U_i is an open disk in $\mathbb{C}_p$.

For each $i = 1, \dots, g$, consider the f_i-orbit of x_i. If x_i is periodic, let $k_i \geq 1$ denote its period, and for every $\ell \geq 0$, define the power series $h_{i,\ell}$ to be the constant $f_i^\ell(x_i)$. Otherwise, choose $k_i \geq 1$ and radii $r_i, s_i > 0$ according to Lemma 3.3, along with the associated conjugating maps $h_{i,\ell}$ for each $\ell \geq 0$.

Let $k = \operatorname{lcm}(k_1, \dots, k_g) \geq 1$. For each $\ell \in \{0, \dots, k-1\}$ such that $V(\mathbb{C}_p) \cap \mathcal{O}_{\Phi^k}(\Phi^\ell(\alpha))$ is finite, we can cover $V(\mathbb{C}_p) \cap \mathcal{O}_{\Phi^k}(\Phi^\ell(\alpha))$ by finitely many singleton orbits.

It remains to consider those $\ell \in \{0, \dots, k-1\}$ for which there is an infinite set $\mathcal{N}$ of nonnegative integers n such that $\Phi^{nk+\ell}(\alpha) \in V(\mathbb{C}_p)$. We will show that in fact, $\Phi^{nk+\ell}(\alpha) \in V(\mathbb{C}_p)$ for *all* $n \in \mathbb{N}$.

For any $|z| \leq 1$, note that $kz \in \overline{D}(0, s_i)$ for all $i = 1, \dots, g$. Thus, it makes sense to define $\theta : \overline{D}(0,1) \to U_1 \times \cdots \times U_g$ by

$$\theta(z) = (h_{1,\ell}(kz), \dots, h_{g,\ell}(kz));$$

Then for all $n \geq 0$, we have

$$\theta(n) = \Phi^{nk+\ell}(\alpha),$$

because for each $i = 1, \dots, g$, we have $k_i | k$, and therefore

$$h_{i,\ell}(nk) = h_{i,\ell}(nk + h_{i,\ell}^{-1}(f_i^\ell(x_i))) = f_i^{nk}(f_i^\ell(x_i)) = f_i^{nk+\ell}(x_i).$$

Given any polynomial F vanishing on V, the composition $F \circ \theta$ is a convergent power series on $\overline{D}(0,1)$ that vanishes at all integers in $\mathcal{N}$. However, a nonzero convergent power series can have only finitely many zeros in $\overline{D}(0,1)$; see, for example, [Rob00, Section 6.2.1]. Thus, $F \circ \theta$ is identically zero. Therefore,

$$F(\Phi^{nk+\ell}(\alpha)) = F(\theta(n)) = 0$$

for all $n \geq 0$, not just $n \in \mathcal{N}$. This is true for all such F, and therefore $\mathcal{O}_{\Phi^k}(\Phi^\ell(\alpha)) \subseteq V(\mathbb{C}_p)$.

The conclusion of Theorem 3.2 now follows, because $\mathcal{O}$ is the finite union of the orbits $\mathcal{O}_{\Phi^k}(\Phi^\ell(\alpha))$ for $0 \leq \ell \leq k-1$. □

As an immediate corollary, we have the following result, which proves Conjecture 1.1 in the case that Φ is defined over $\overline{\mathbb{Q}}$ and there is a nonarchimedean place v with the following property: for each i, the rational function f_i has good reduction at v, and $\mathcal{O}_{f_i}(x_i)$ avoids all v-adic attracting periodic points.

Theorem 3.4. *Let V be a subvariety of $\left(\mathbb{P}^1\right)^g$ defined over $\mathbb{C}_p$, let $f_1, \ldots, f_g \in \mathbb{C}_p(t)$ be rational functions of good reduction on $\mathbb{P}^1$, and let Φ denote the coordinatewise action of $(f_1, \ldots, f_g)$ on $\left(\mathbb{P}^1\right)^g$. Let $\mathcal{O}$ be the Φ-orbit of a point $\alpha = (x_1, \ldots, x_g) \in (\mathbb{P}^1(\mathbb{C}_p))^g$, and suppose that for each i, the orbit $\mathcal{O}_{f_i}(x_i)$ does not intersect the residue class of any attracting f_i-periodic point. Then $V(\mathbb{C}_p) \cap \mathcal{O}$ is a union of at most finitely many orbits of the form $\{\Phi^{nk+\ell}(\alpha)\}_{n \geq 0}$ for nonnegative integers k and ℓ.*

Proof. For each i, the reduction $r_p(x_i) \in \mathbb{P}^1(\overline{\mathbb{F}_p})$ is preperiodic under the reduced map $(f_i)_p$. Replacing α by $\Phi^m(\alpha)$ for some $m \geq 0$, and replacing Φ by Φ^j for some $j \geq 1$, then, we may assume that for each i, the residue class U_i of x_i is mapped to itself by f_i. By hypothesis, there are no attracting periodic points in those residue classes; thus, by [RL03, Proposition 4.32] (for example), U_i is a quasiperiodicity disk for f_i. Theorem 3.2 now yields the desired conclusion. □

4. Preliminary results on intersection theory

In this section we prove the following result on intersection theory for arithmetic surfaces. It will be used in the proofs of our main results in Section 5.

Theorem 4.1. *Let K be a number field, and let $\varphi : \mathbb{P}^1_K \longrightarrow \mathbb{P}^1_K$ be a morphism defined over K of degree at least 2 that is not conjugate to a map of the form $t \mapsto t^n$ for any integer n. Suppose φ does not have any superattracting periodic points other than exceptional points.*

Let $\alpha, \beta \in \mathbb{P}^1(K)$ be points that are not preperiodic for φ. Suppose that there is a curve $C \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ such that there are infinitely many integers $k \geq 0$ for which $\varphi^k(\alpha, \beta) \in C(K)$. Then there are infinitely many finite places v of K such that φ has good reduction at v and such that for some integer $n \geq 1$, the points $\varphi^n(\alpha)$ and $\varphi^n(\beta)$ are in the same residue class at the place v; i.e., $r_v(\varphi^n(\alpha)) = r_v(\varphi^n(\beta))$.

The condition on superattracting points is equivalent to stipulating that the nonexceptional critical points of φ are not periodic. Note that φ has at most one exceptional point, since it is not conjugate to $t \mapsto t^n$.

By [MS95, Proposition 4.2], φ has good reduction at all but finitely many places v of K. (See Section 2 for a discussion of good reduction and the reduction map r_v.) Thus, the content of Theorem 4.1 is the common reduction of $\varphi^n(\alpha)$ and $\varphi^n(\beta)$.

Before proving the Theorem, we set some notation. Let V be a variety over a number field K, and let $\mathcal{V}$ be a model for V over the ring of integers $\mathfrak{o}_K$ of K. Let S be a finite set of places of K that contains all of the archimedean places of K, and let Z be an effective divisor on V. We say that a point γ on V is *S-integral for Z* if the Zariski closure of γ does not meet the Zariski closure of $\operatorname{Supp} Z$ in $\mathcal{V}$ at any fibres of $\mathcal{V}$ outside of S.

More specifically, let $\mathcal{V}$ be the model $\mathbb{P}^1_{\mathfrak{o}_K} \times \mathbb{P}^1_{\mathfrak{o}_K}$ for $\mathbb{P}^1_K \times \mathbb{P}^1_K$ that comes from the isomorphism between $\mathbb{P}^1_K$ and the generic fibre of $\mathbb{P}^1_{\mathfrak{o}_K}$ we chose in Section 2. We will say that a point Q on $\mathbb{P}^1_K \times \mathbb{P}^1_K$ is *S-integral* for a divisor Z if it is S-integral for Z with respect to $\mathcal{V}$.

Let
$$\Phi : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$$
be the map $\Phi = \varphi \times \varphi$, and let Δ denote the diagonal divisor on $\mathbb{P}^1 \times \mathbb{P}^1$. We will need the following proposition.

Proposition 4.2. *Let $\varphi : \mathbb{P}^1_K \longrightarrow \mathbb{P}^1_K$ be a rational map of degree $d > 1$ that has no periodic critical points. Let α and β be points in $\mathbb{P}^1(K)$ that are not preperiodic for φ, and let S be a finite set of places of K that contains all of the archimedean places of K. Let C be a curve in $\mathbb{P}^1 \times \mathbb{P}^1$. Then there are at most finitely many integers $k \geq 0$ that satisfy both of the following conditions:*

(i) $\Phi^k(\alpha, \beta) \in C$*; and*
(ii) $\Phi^k(\alpha, \beta)$ *is S-integral for Δ.*

For any nonconstant morphism $h : \mathbb{P}^1 \to \mathbb{P}^1$ and any point x on $\mathbb{P}^1$, we will denote the ramification index of x over $h(x)$ by $e(x/h(x))$.

We will need the following result about ramification.

Lemma 4.3. *Let $h : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ be a nonconstant morphism defined over a field K of characteristic 0, and let $H := (h, h)$ its action coordinatewise on $\mathbb{P}^1 \times \mathbb{P}^1$. Then:*

(i) *For any point $(P, Q) \in \mathbb{P}^1(K) \times \mathbb{P}^1(K)$, the multiplicity of $\Delta_H := H^*(\Delta)$ at (P, Q) is at most $\max_{x \in \mathbb{P}^1} e(x/h(x))$.*
(ii) *Each irreducible component of Δ_H has multiplicity one.*

Proof. By performing the same change of coordinates on both copies of $\mathbb{P}^1$, we may assume that the point at infinity is not among the points $P, h(P), Q, h(Q)$. Hence, let $t_0, u_0 \in K$ such that $P = [t_0 : 1]$, and $Q = [u_0 : 1]$. Then h has a local power series expansion (see [Sha77, II.2]) in a neighborhood of P as $h(t) = a_0 + \sum_{i=e_1}^{\infty} a_i(t - t_0)^i$ and in a neighborhood of Q as $h(u) = b_0 + \sum_{i=e_2}^{\infty} b_i(u - u_0)^i$, where $a_i, b_i \in K$, and $e_1 \geq 1$ and $e_2 \geq 1$ are the ramification indices of h at P and Q, respectively. Clearly, $(P, Q) \in \Delta_H$ if and only if $a_0 = b_0$. Thus, we may assume $a_0 = b_0$, and so, near (P, Q), the subvariety Δ_H is defined by the equation
$$\sum_{i=e_1}^{\infty} a_i(t - t_0)^i - \sum_{i=e_2}^{\infty} b_i(u - u_0)^i = 0.$$
The multiplicity of (P, Q) as a point on Δ_H is therefore given by $\min(e_1, e_2)$ (see [Sha77, IV.1]); since $e_1, e_2 \leq \max_{x \in \mathbb{P}^1} e(x/h(x))$, statement (i) follows.

Moreover, Δ_H has multiplicity more than one at (P, Q) only if h is ramified at both P and Q. Because h is ramified at only finitely many points of $\mathbb{P}^1$ (note that $\mathrm{char}(K) = 0$), there are at most finitely many points

$(P, Q) \in \mathbb{P}^1 \times \mathbb{P}^1$ at which Δ_H has multiplicity larger than one, proving statement (ii). □

We set more notation, as follows. For each $n \geq 0$, let X_n be the divisor $(\Phi^n)^*(\Delta)$. Note that $\Delta \subseteq X_n$ for each n. Therefore, more generally, for each $0 \leq m < n$, we have $X_m \subseteq X_n$. Let $Y_0 := X_0 = \Delta$, and for $n \geq 1$, let

$$Y_n = (\Phi^n)^*(\Delta) - (\Phi^{n-1})^*(\Delta).$$

Then we have

$$X_n = \bigcup_{i=0}^{n} Y_i.$$

Note that Y_n is nonempty because $\deg(\Phi) > 1$. Furthermore, by Lemma 4.3, each irreducible component of X_n, and hence of Y_n, has multiplicity one.

We also have the following important result, giving a uniform bound for the ramification of φ^n.

Lemma 4.4. *Let $\varphi : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ be a map which has no periodic critical points. Let $Q_1, \dots, Q_m$ be the ramification points of φ. Then for any n and any point P on $\mathbb{P}^1$,*

$$e(P/\varphi^n(P)) \leq \prod_{i=1}^{m} e(Q_i/\varphi(Q_i)). \tag{4.4.1}$$

Proof. For each integer $i = 1, \dots, m$, there is at most one $j \geq 0$ such that $\varphi^j(P) = Q_i$, since none of the Q_i are periodic. Meanwhile, $e(\varphi^j(P)/\varphi^{j+1}(P))$ equals 1 for all $j \geq 0$ such that $\varphi^j(P)$ is not a ramification point. Thus,

$$e(P/\varphi^n(P)) = \prod_{j=0}^{n-1} e(\varphi^j(P)/\varphi^{j+1}(P)) \leq \prod_{i=1}^{m} e(Q_i/\varphi(Q_i)).$$ □

Combining Lemmas 4.3 and 4.4 gives the following result.

Lemma 4.5. *Under the hypothesis of Lemma 4.4 and with the above notation for X_n and Y_n, there is a constant $M \geq 0$ such that for any point $Q \in \mathbb{P}^1(\overline{K}) \times \mathbb{P}^1(\overline{K})$, at most M of the Y_n contain Q.*

Proof. Let M be the quantity on the right hand side of (4.4.1). If a point Q is contained in $M + 1$ different Y_i, then the multiplicity of Q on X_n is at least $M + 1$ for some large enough n. Lemma 4.4 and part (i) of Lemma 4.3 now give a contradiction. □

We are now ready to prove Proposition 4.2.

Proof of Proposition 4.2. Enlarge S if necessary to contain not only all archimedean places of K but also all places of bad reduction for Φ. Fix an integer $n \geq 2M$, where M is as in Lemma 4.5.

Given an irreducible curve E in $\mathbb{P}^1 \times \mathbb{P}^1$ that does not map to a point under either of the projection maps on $\mathbb{P}^1 \times \mathbb{P}^1$, we claim that E intersects X_n in at least three distinct points. Indeed, E must meet Y_m in at least one point for all $m \geq 0$. However, for each point Q, at most M of the Y_m (for $0 \leq m \leq n$) contain Q. Hence, E must intersect X_n in at least three distinct points, as desired.

Now suppose that there are infinitely many k (and hence infinitely many $k > n$) such that $\Phi^k(\alpha, \beta) \in C$ and $\Phi^k(\alpha, \beta)$ is S-integral for Δ. For each such $k > n$, then, $\Phi^{k-n}(\alpha, \beta)$ is S-integral for X_n. Then there is a K-irreducible curve Z in $(\Phi^n)^{-1}(C)$ such that there are infinitely many m for which $\Phi^m(\alpha, \beta) \in Z$ and $\Phi^m(\alpha, \beta)$ is S-integral for X_n. Because α and β are not preperiodic, Z does not project to a single point on either of the two coordinates of $\mathbb{P}^1 \times \mathbb{P}^1$. In addition, Z contains infinitely many K-rational points $\Phi^m(\alpha, \beta)$; because it is also irreducible over K, it is in fact geometrically irreducible (note that if a component of Z defined over a finite extension of K has infinitely many K-rational points, then it is in fact defined over K.)

Thus, by our claim, X_n meets Z in at least three points. However, Z contains infinitely many points $\Phi^m(\alpha, \beta)$ that are S-integral for X_n; this is impossible, by Siegel's theorem on integral points. □

We treat the case of polynomials (which *do* have a periodic critical point) slightly differently. For a polynomial $f(t) = \sum_{i=0}^d a_i t^i$ with $a_d \neq 0$, we define its homogenization $F(t, u)$ by $F(t, u) = \sum_{i=0}^d a_i t^i u^{d-i}$. We then define $\Phi_f : \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ by

$$\Phi_f([x : y : z]) = [F(x, z) : F(y, z) : z^d].$$

Let D be the divisor on $\mathbb{P}^2$ consisting of all points $[x : y : z]$ such that $x = y$. The divisor D will play the same role here that the diagonal Δ played on $\mathbb{P}^1 \times \mathbb{P}^1$. We let

$$A_n = (\Phi_f^n)^*(D), \quad \text{and} \quad B_n = (\Phi_f^n)^*(D) - (\Phi_f^{n-1})^*(D).$$

Then

$$A_n = \bigcup_{i=0}^n B_i.$$

Let $\mathcal{W}$ be the model $\mathbb{P}^2_{\mathfrak{o}_K}$ for $\mathbb{P}^2_K$. We will say that a point Q on $\mathbb{P}^2_K$ is S-integral for a divisor Z if it is S-integral for Z with respect to $\mathcal{W}$.

Proposition 4.6. *Let $f \in K[t]$ be a polynomial with no periodic critical points other than the point at infinity. Let α and β be points in $\mathbb{A}^1(K)$ that are not preperiodic for f, and let S be a finite set of places of K that contains all of the archimedean places. Let C be a curve in $\mathbb{P}^2$. Then there are at most finitely many k that satisfy both of the following conditions:*

(i) $\Phi_f^k([\alpha : \beta : 1]) \in C$*; and*
(ii) $\Phi_f^k([\alpha : \beta : 1])$ *is S-integral for D.*

Proof. The proof is almost identical to the proof of Proposition 4.2. Note that if $[x : y : 0]$ lies on A_n, then $[x : y : 0]$ must be in the inverse image of

$[1:1:0]$ under Φ_f, which is equivalent to saying that $x^{d^n} = y^{d^n}$. There are exactly d^n such points, and d^n is also the degree of A_n; so each point of the form $[x:y:0]$ must have multiplicity one on A_n (and hence on B_n) for any n. Then, as in Proposition 4.2, we can bound the multiplicity of any point $[x:y:1]$ on A_n by

$$\Big(\prod_{i=1}^{m} e(Q_i/f(Q_i))\Big)$$

where the Q_i are the ramification points of f other than infinity. Thus, again, if there are infinitely many points on C that are S-integral for D, then for any n, there are infinitely many points on some irreducible curve E in $(\Phi_f^n)^{-1}(C)$ that are S-integral for A_n. When n is at least $2 \cdot \prod_{i=1}^{m} e(Q_i/f(Q_i))$, such a curve E must meet A_n in at least three distinct points, which gives us a contradiction by Siegel's theorem. □

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. If $\varphi : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ has no periodic critical points, Proposition 4.2 implies that for any finite set S of places of K, there are only finitely many n such that $\varphi^n(\alpha)$ does not meet $\varphi^n(\beta)$ at any v outside of S. Thus, there must be infinitely many places v such that $r_v(\varphi^n(\alpha)) = r_v(\varphi^n(\beta))$ for some $n \in \mathbb{N}$.

On the other hand, if φ has an exceptional point, then after changing coordinates, we have $\varphi = f$ for some polynomial f (note that φ does not have two exceptional points, as it is not conjugate to a map of the form $t \mapsto t^n$). Furthermore, since φ has no non-exceptional periodic critical points, it follows that f has no periodic critical points save the point at infinity. By Proposition 4.6, for any finite set S of places of K, there are at most finitely many n such that $f^n(\alpha) - f^n(\beta)$ is an S-unit. Thus, there are infinitely many places v such that $r_v(f^n(\alpha)) = r_v(f^n(\beta))$ for some $n \in \mathbb{N}$. □

5. Dynamical Mordell-Lang for curves

Using Theorem 4.1 we can prove a dynamical Mordell-Lang statement for curves embedded in $\mathbb{P}^1 \times \mathbb{P}^1$.

Theorem 5.1. *Let $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a curve defined over $\overline{\mathbb{Q}}$, and let $\Phi := (\varphi, \varphi)$ act on $\mathbb{P}^1 \times \mathbb{P}^1$, where $\varphi \in \overline{\mathbb{Q}}(t)$ is a rational function with no superattracting periodic points other than exceptional points. Let $\mathcal{O}$ be the Φ-orbit of a point $(x,y) \in (\mathbb{P}^1 \times \mathbb{P}^1)(\overline{\mathbb{Q}})$. Then $C(\overline{\mathbb{Q}}) \cap \mathcal{O}$ is a union of at most finitely many orbits of the form $\{\Phi^{nk+\ell}(x,y)\}_{n \geq 0}$ for $k, \ell \in \mathbb{N}$.*

Proof. When $\deg(\varphi) = 1$, the result follows immediately from work of Denis [Den94] and Bell [Bel06], since in this case Φ induces an automorphism of $\mathbb{A}^2$. Hence, we may assume $\deg(\varphi) \geq 2$.

If φ has two exceptional points, then φ is conjugate to the map $t \mapsto t^n$, for some $n \in \mathbb{Z}$. Then our result follows from [GTa, Theorem 1.2], as Φ

induces an endomorphism of $\mathbb{G}_m^2$. Thus, we may assume that φ has at most one exceptional point, and no other periodic critical points.

We may assume that C is irreducible, and that $C(\overline{\mathbb{Q}}) \cap \mathcal{O}$ is infinite. We may also assume that neither x nor y is φ-preperiodic, because in that case the projection of C to one of the two coordinates of $\mathbb{P}^1 \times \mathbb{P}^1$ consists of a single point (which would be a φ-periodic point), and the conclusion of Theorem 5.1 would be immediate.

Let K be a number field over which φ, C, and (x, y) are defined. By the previous paragraph, the hypotheses of Theorem 4.1 hold for $(\alpha, \beta) = (x, y)$. Thus, there are infinitely many nonarchimedean places v of K at which φ has good reduction and such that $r_v(\varphi^n(x)) = r_v(\varphi^n(y))$ for some integer $n \geq 1$. Fix such a place v.

Let $p \in \mathbb{N}$ be the prime number lying in the maximal ideal of the nonarchimedean place v, fix an embedding of K into $\mathbb{C}_p$ respecting v, and let U denote the residue class of $\mathbb{P}^1(\mathbb{C}_p)$ containing $\varphi^n(x)$ and $\varphi^n(y)$. Since φ has good reduction, every iterate $\varphi^{n+k}(U)$ is a residue class, and it contains both $\varphi^{n+k}(x)$ and $\varphi^{n+k}(y)$. If no such residue class contains an attracting periodic point, then our desired conclusion is immediate from Theorem 3.4.

The remaining case is that some residue class $\varphi^{n+k}(U)$ contains an attracting periodic point, which must therefore attract the orbits of both x and y. The Theorem now follows from [GTb, Theorem 2.2] (and from Fact 3.5 of the same paper, specifying the radius R). $\square$

We can now prove Theorem 1.3 as a consequence of Theorem 5.1.

Proof of Theorem 1.3. We may assume that C is irreducible, and that $C(\overline{\mathbb{Q}}) \cap \mathcal{O}$ is infinite. It suffices to prove that C is Φ-periodic. Indeed, if $\Phi^k(C) = C$, then for each $\ell \in \{0, \ldots, k-1\}$, the intersection of C with $\mathcal{O}_{\Phi^k}(\Phi^\ell(\alpha))$ either is empty or else consists of all $\Phi^{kn+\ell}(\alpha)$, for some n sufficiently large. Either way, the conclusion of Theorem 1.3 holds.

We argue by induction on g. The case $g = 1$ is obvious, while the case $g = 2$ is proved in Theorem 5.1. Assuming Theorem 1.3 for some $g \geq 2$, we will now prove it for $g+1$. We may assume that C projects dominantly onto each of the coordinates of $\left(\mathbb{P}^1\right)^{g+1}$; otherwise, we may view C as a curve in $\left(\mathbb{P}^1\right)^g$, and apply the inductive hypothesis. We may also assume that no x_i is preperiodic, lest C should fail to project dominantly on the i^{th} coordinate.

Let $\pi_1 : \left(\mathbb{P}^1\right)^{g+1} \to \left(\mathbb{P}^1\right)^g$ be the projection onto the first g coordinates, let $C_1 := \pi_1(C)$, and let $\mathcal{O}_1 := \pi_1(\mathcal{O})$. By our assumptions, C_1 is an irreducible curve that has an infinite intersection with $\mathcal{O}_1$. By the inductive hypothesis, C_1 is periodic under the coordinatewise action of φ on the first g coordinates of $\left(\mathbb{P}^1\right)^{g+1}$.

Similarly, let C_2 be the projection of C on the last g coordinates of $\left(\mathbb{P}^1\right)^{g+1}$. By the same argument, C_2 is periodic under the coordinatewise action of φ on the last g coordinates of $\left(\mathbb{P}^1\right)^{g+1}$.

Thus, C is Φ-preperiodic, because it is an irreducible component of the one-dimensional variety $\left(C_1 \times \mathbb{P}^1\right) \cap \left(\mathbb{P}^1 \times C_2\right)$, and because both $C_1 \times \mathbb{P}^1$ and $\mathbb{P}^1 \times C_2$ are Φ-periodic.

Claim 5.2. *Let X be a variety, let $\alpha \in X(\overline{K})$, let $\Phi : X \longrightarrow X$ be a morphism, and let $C \subset X$ be an irreducible curve that has infinite intersection with the orbit $\mathcal{O}_\Phi(\alpha)$. If C is Φ-preperiodic, then C is Φ-periodic.*

Proof of Claim 5.2. Assume C is not periodic. Because C is preperiodic, there exist $k_0, n_0 \geq 1$ such that $\Phi^{n_0}(C)$ is periodic of period k_0. Let $k := n_0 k_0$, and let $C' := \Phi^k(C)$, which is fixed by Φ^k. Then $C \neq C'$, since C is not periodic. Because C and C' are irreducible curves, it follows that

$$C \cap C' \text{ is finite.} \tag{5.2.1}$$

On the other hand, there exists $\ell \in \{0, \ldots, k-1\}$ such that $C \cap \mathcal{O}_{\Phi^k}(\Phi^\ell(\alpha))$ is infinite, because $C \cap \mathcal{O}_\Phi(\alpha)$ is infinite. Let $n_1 \in \mathbb{N}$ be the smallest nonnegative integer n such that $\Phi^{nk+\ell}(\alpha) \in C$. Since $C' = \Phi^k(C)$ is fixed by Φ^k, we conclude that $\Phi^{nk+\ell}(\alpha) \in C'$ for each $n \geq n_1 + 1$. Therefore

$$C \cap \mathcal{O}_{\Phi^k}(\Phi^\ell(\alpha)) \cap C' \text{ is infinite.} \tag{5.2.2}$$

Statements (5.2.1) and (5.2.2) are contradictory, proving the claim. ☐

An application of Claim 5.2 with $X = (\mathbb{P}^1)^{g+1}$ now completes the proof of Theorem 1.3. ☐

6. Quadratic polynomials

In this Section, we will prove Theorems 1.4 and 1.5. We will continue to work with the same reduction maps $r_v : \mathbb{P}^1(K) \longrightarrow \mathbb{P}^1(k_v)$ as in Section 2, where v is a finite place of K. We begin with a lemma derived from work of Silverman [Sil93].

Lemma 6.1. *Let $\varphi : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ be a morphism of degree greater than one, let $\alpha \in \mathbb{P}^1(K)$ be a point that is not preperiodic for φ, and let $\beta \in \mathbb{P}^1(K)$ be a nonexceptional point. Then there are infinitely many v such that there is some positive integer n for which $r_v(\varphi^n(\alpha)) = r_v(\beta)$.*

Proof. Suppose there were only finitely many such v; let S be the set of all such v, together with all the archimedean places. We may choose coordinates $[x : y]$ for $\mathbb{P}^1_K$ such that β is the point $[1 : 0]$. Since $[1 : 0]$ is not exceptional for φ, we see that φ^2 is not a polynomial with respect to this coordinate system. Therefore, by [Sil93, Theorem 2.2], there are at most finitely many n such that $\varphi^n(\alpha) = [t : 1]$ for $t \in \mathfrak{o}_S$, where $\mathfrak{o}_S$ is the ring of S-integers in K. Hence, for all but finitely many integers $n \geq 0$, there is some $v \notin S$ such that $r_v(\varphi^n(\alpha)) = r_v(\beta)$; but this contradicts our original supposition. ☐

Recall that if f has good reduction at a finite place v of K, we write f_v for the reduction of f at v.

Lemma 6.2. *Let $\varphi : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ be a morphism of degree greater than one, and let $\alpha \in K$ be a point that is not periodic for φ. Then there are infinitely many places v of good reduction for φ such that $r_v(\alpha)$ is not periodic for φ_v.*

Proof. If α is φ-preperiodic but not periodic, then the φ-orbit $\mathcal{O}_\varphi(\alpha)$ is finite. Hence, the reduction map r_v is injective on $\mathcal{O}_\varphi(\alpha)$ for all but finitely many places v, and Lemma 6.2 holds in this case.

Thus, we may assume that α is not preperiodic. After passing to a finite extension L of K, we may also assume that φ has a nonexceptional fixed point β. We extend our isomorphism between $\mathbb{P}^1_K$ and the generic fibre of $\mathbb{P}^1_{\mathfrak{o}_K}$ to an isomorphism from $\mathbb{P}^1_L$ to the generic fibre of $\mathbb{P}^1_{\mathfrak{o}_L}$; and for each place $w|v$ of L, we obtain reduction maps $r_w : \mathbb{P}^1(L) \longrightarrow \mathbb{P}^1(\ell_w)$, where ℓ_w is the residue field at w. For each such $w|v$, we have $r_v(\gamma) = r_w(\gamma)$ for any $\gamma \in \mathbb{P}^1(K)$. By Lemma 6.1, there are infinitely many places w such that there is some n for which $r_w(\varphi^n(\alpha)) = r_w(\beta)$. When $w|v$ for v a place of good reduction for φ, this means that $r_v(\varphi^m(\alpha)) = r_v(\varphi^n(\alpha)) = r_w(\beta)$ for all $m \geq n$, since β is fixed by φ. At all but finitely many of these v, we have $r_v(\alpha) \neq r_w(\beta)$, which means that there is no positive integer m such that $r_v(\varphi^m(\alpha)) = r_v(\alpha)$, as desired. □

We also need the following result for quadratic polynomials.

Proposition 6.3. *Let K be a number field, and let $f \in K[t]$ be a quadratic polynomial with no periodic critical points other than the point at infinity. Then there are infinitely many finite places v of K such that $|f'(z)|_v = 1$ for each $z \in K$ such that $|z|_v \leq 1$ and $r_v(z)$ is f_v-periodic.*

Proof. Since f is a quadratic polynomial, it only has one critical point α other than the point at infinity. By Lemma 6.2 and because α is not periodic, there are infinitely many places v of good reduction for f such that $r_v(\alpha)$ is not f_v-periodic, and such that $|\alpha|_v \leq 1$ and $|2|_v = 1$. (The last two conditions may be added because each excludes only finitely many v.) In particular, $|f'(z)|_v = |z - \alpha|_v$ for any $z \in K$.

Hence, for any such v, and for any $z \in K$ as in the hypotheses, we have $r_v(z) \neq r_v(\alpha)$, since $r_v(z)$ is periodic but $r_v(\alpha)$ is not. Thus, $|f'(z)|_v = |z - \alpha|_v = 1$. □

We are now ready to prove Theorems 1.4 and 1.5.

Proof of Theorem 1.4. Let K be a number field such that V is defined over K, the polynomial f is in $K[t]$, and $x_1, \ldots, x_g$ are all in K.

Using Proposition 6.3, we may choose a place v of K such that

(a) v is a place of good reduction for f;
(b) $|x_i|_v \leq 1$, for each $i = 1, \ldots, g$;
(c) $|f'(z)|_v = 1$ for all z such that $|z|_v \leq 1$ and $r_v(z)$ is f_v-periodic.

Indeed, conditions (a) and (b) are satisfied at all but finitely many places v, while condition (c) is satisfied at infinitely many places. Because f is a polynomial, conditions (a) and (b) together imply that $|f^n(x_i)|_v \leq 1$ for all $i = 1, \ldots, g$ and $n \geq 0$. Meanwhile, condition (c) implies that f has no attracting periodic points at v. The desired conclusion now follows from Theorem 3.4. □

Proof of Theorem 1.5. After changing coordinates, we assume that $f(t) = t^2 + c$ for some $c \in \mathbb{Q}$. Thus, 0 is the only finite critical point of f. If $c \notin \mathbb{Z}$, then there is some p such that $|c|_p > 1$. But then $|f^n(0)|_p \to \infty$, so 0 cannot be periodic. Similarly, if c is an integer other than 0, -1 or -2, then we have $|f^n(0)|_\infty \to \infty$, so 0 cannot be periodic. If $c = -2$, then 0 is only f-preperiodic, but not f-periodic. In all the above cases, the hypotheses of Theorem 1.4 are met, and our proof is done. If $c = 0$, then $f(t) = t^2$ is an endomorphism of $\mathbb{G}_m^g$, and thus our result follows from [GTa, Theorem 1.2].

We are left with the case that $f(t) = t^2 - 1$. As in the proof of Theorem 1.3, we may assume (via induction on g) that no x_i is preperiodic; in particular, all x_i and $f(x_i)$ are nonzero. If $f^2(z) = 0$, then either $z = 0$, or $z = \pm\sqrt{2}$. Bearing this fact in mind, we note that there are infinitely many primes p such that 2 is not a quadratic residue modulo p. Thus, we may choose an odd prime p such that each x_i and $f(x_i)$ is a p-adic unit, and such that 2 is not a quadratic residue modulo p. Then there is no positive integer n such that $f^n(x_i)$ is in the same residue class as 0 modulo p for any i. Therefore, $|f'(f^n(x_i))|_p = 1$ for all n, and hence $f^n(x_i)$ never lies in the same residue class as an attracting periodic point. Theorem 1.5 now follows from Theorem 3.4. □

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