Take Home Midterm

1. For each of the following rings, explain whether or not the ring integrally closed:

(a) The polynomial right $\mathbb{Q}[t]$.
(b) The ring $\mathbb{Z}[\sqrt[5]{2}]$.
(c) The ring $\mathbb{Z}[\sqrt[3]{37}]$.
(d) The ring $\mathbb{Z}[i, \sqrt{3}]$.

2. In an earlier homework, you should that for any $\alpha \in \mathbb{C}$, there is an element β of the right $\mathbb{Z}[\frac{1+\sqrt{-11}}{2}]$ such that $|\alpha - \beta| < 1$ where $|\cdot|$ is the usual norm on $\mathbb{C}$ given by $|a+ib| = \sqrt{a^2+b^2}$. In this problem we examine whether or not this is true for some other additive subgroups of $\mathbb{C}$, which do not have a ring structure.

(a) Let

$$\mathcal{L}_1 = \{a + b\frac{1+\sqrt{-12}}{2} \mid a, b \in \mathbb{Z}\}.$$

True or false and explain: for every $\alpha \in \mathbb{C}$, there is a $\beta \in \mathcal{L}_1$ such that $|\alpha - \beta| < 1$.

(b) Let

$$\mathcal{L}_2 = \{a + b\frac{1+\sqrt{-14}}{2} \mid a, b \in \mathbb{Z}\}.$$

True or false and explain: for every $\alpha \in \mathbb{C}$, there is a $\beta \in \mathcal{L}_2$ such that $|\alpha - \beta| < 1$.

3. For each of the following, explain why the statement is **True** or give a counterexample showing that it is **False**. In all of the following R is a ring (commutative with identity).

(a) If R is a Noetherian ring and I is a proper ideal of R, then R/I is a Noetherian ring.
(b) If I is a proper ideal of R and R/I is a Noetherian ring, then R is a Noetherian ring.
(c) If R/a is a Noetherian ring for all nonzero $a \in R$, then R is a Noetherian ring.
(d) If R is a Noetherian ring and $\alpha_1, \ldots, \alpha_m$ are integral over R in some extension $R \subseteq B$, then $R[\alpha_1, \ldots, \alpha_m]$ is a Notherian ring.

4. For each of the following rings R, find all nonzero primes $\mathfrak{q}$ such that $R_{\mathfrak{q}}\mathfrak{q}$ is not principal (you can write these as $\mathfrak{q} = (p, g_i(\alpha))$ for $R = \mathbb{Z}[\alpha]$, as in class).

(a) $\mathbb{Z}[\sqrt{27}]$.
(b) $\mathbb{Z}[\sqrt{5}]$.

(c) $\mathbb{Z}[\sqrt[3]{19}]$

5. Let R be an integral domain with field of fractions K. Let $I \subseteq K$ have the property that $aI \subseteq I$ for all $a \in R$; in other words, I is an R-submodule or K. As in class we define

$$(R : I) = \{x \in K \mid xI \subseteq R\}.$$

(a) Show that if I is finitely generated, then $(R : I) \neq \{0\}$.

(b) Show (via an example) that there are R and I such that $(R : I) = \{0\}$.

6. Problem 6 on Page 58.
7. Problem 7 on Page 58
8. Problem 2 on p. 62.
9. Let L be a degree n field extension of $\mathbb{Q}$. Let $B \subset L$ be a ring that is integral over $\mathbb{Z}$ and has field of fractions L. Let $\sigma_1, \ldots, \sigma_n$ be the n distinct embeddings $\sigma : L \longrightarrow \mathbb{C}$. Show that for any basis $w_1, \ldots, w_n$ for B as an A-module, we have

$$\Delta(B/\mathbb{Z}) = \left(\det[\sigma_i(w_j)]\right)^2.$$

[Hint: Multiply $[\sigma_i(w_j)]$ by its transpose and use the fact (that you should prove) that $\mathrm{T}_{L/K}(y) = \sigma_1(y) + \cdots + \sigma_n(y)$ for any $y \in L$.]

10. Let R_K be Dedekind domain with field of fractions K. Let E and L be finite separable extensions of K, of degree m and n, respectively, and let R_E and R_L be the integral closures of R_K in E and L respectively. Suppose that $\mathcal{P}$ ramifies completely in R_E, i. e. that $\mathcal{P}R_E = \mathcal{Q}^m$. Suppose also that $\mathcal{P}R_L = \mathcal{M}_1^{e_1} \cdots \mathcal{M}^{e_t}$ where $\gcd(e_i, m) = 1$ for some e_i. Show that E and L are linearly disjoint over K. [Hint: factor $\mathcal{P}$ in the integral closure of R_K in $E \cdot L$.]

11. Let p be a prime and let a be a positive integer that is not a perfect p-th power.

(a) Show that $x^p - a$ is irreducible over $\mathbb{Z}$.

(b) Let α be a root of $x^p - a$ and let ξ_p be a primitive p-th root of unity. Show that $\mathbb{Q}(\xi_p)$ and $\mathbb{Q}(\alpha)$ are linearly disjoint over $\mathbb{Q}$.

(c) Let K be the splitting field of $x^p - a$ over $\mathbb{Q}$. Describe its Galois group over $\mathbb{Q}$ (as a semidirect product of abelian groups).]

12. Let m be a positive integer and let a be an integer with at least one prime factor p such that p^2 doesn't divide a and p doesn't divide m. Show that $\mathbb{Q}(\xi_m)$ and $\mathbb{Q}(\sqrt[m]{a})$ are linearly disjoint.