Math 568 Problem Set #7 Due 11/3/2014

1. Let a be a square-free positive integer. Let p be a prime number. Show that $\mathbb{Z}[\sqrt[p]{a}]$ is integrally closed if and only if $a^p - a \not\equiv 0 \pmod{p^2}$.

2. Let $(\cdot,\cdot)$ be a symmetric bilinear form on a vector space V over a field K. Let $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$ be bases for V over K. Let M be the matrix whose ij-th coefficient is (v_i, v_j). For each w_j, we write $w_j = \sum_{i=1}^m d_{ij} v_i$, and let D be the matrix whose ij-th coefficient is d_{ij}. Let N be the matrix whose ij-th coefficient is (w_i, w_j). Show that

$$N = D^t M D.$$

3. Let D be an $n \times n$ matrix with coefficients in an integral domain R. Show that if $\det D$ is a unit in R, then there is an $n \times n$ matrix E with coefficients in R such that $DE = ED = I_n$ (where I_n is the identity $n \times n$ matrix). [Hint: You can use the classical adjoint.]

4. Let A be a DVR with maximal ideal $\mathcal{P}$ and field of fractions K. Let $B_1 \subseteq B_2$ be integral extensions of A such that the B_1 and B_2 have field of fractions equal to L, where L is a finite separable extension of K. Show that $\Delta(B_1/A) = \mathcal{P}^{2r}\Delta(B_2/A)$, for some $r \geq 0$. Then show that $r = 0$ if and only if $B_1 = B_2$. [Hint:Use problems 2 and 3.]

5. Janusz p.42, Ex. 4.

6. Let A be a Dedekind domain with field of fractions K. Let L and L' be finite separable extensions of K and suppose that there exist $\alpha \in L$ and $\alpha' \in L'$ such that the integral closure of A in L is $A[\alpha]$ and the integral closure of A in L' is $A[\alpha']$. Suppose furthermore that $\Delta(A[\alpha]/A) + \Delta(A[\alpha']/A) = A$ (as ideals). Let M be the compositum LL' over K. Is the integral closure of A in M necessarily equal to $A[\alpha, \alpha']$? Give a proof or a counterexample. [Hint: Use criterion from class on when primes are invertible.]

7. Let p and q be primes in $\mathbb{Z}$ with $p \neq q$. Find the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\xi_{pq})$ where ξ_{pq} is a primitive pq-th root of unity. Justify your answer.

8. Let ξ_{p^n} be a primitive p^n-th root of unity for $n \geq 1$. Calculate $\mathrm{N}_{\mathbb{Q}(\xi_{p^n})/\mathbb{Q}}(1 - \xi_{p^n})$.