Math 568 Problem Set #5 – Due October 20

1. Let R be a domain such that:

(1) For every nonzero prime $\mathfrak{p}$ of R, the localization $R_{\mathfrak{p}}$ is a DVR; and

(2) Every nonzero ideal I of R is contained in at most finitely many primes $\mathfrak{p}$ of R.

Show that R must be a Dedekind domain. [Hint: It is sufficient to show it is Noetherian, for example.]

2. Let K be a field and let L be a totally inseparable extension of K such that $[L : K] = q$. Let A be a Dedekind domain whose field of fractions is A and let B be the integral closure of A in L. Show that for each prime $\mathfrak{p}$ of A there is a unique prime $\mathfrak{q}$ of B such that $\mathfrak{q} \cap A = \mathfrak{p}$. [Consider the map $\varphi : B \longrightarrow A$ given by $\varphi(x) = x^q$ and show that for any prime $\mathfrak{p}$ in A we have $\varphi^{-1}(\mathfrak{p}) \cap A = \mathfrak{p}$ and for any prime $\mathfrak{q}$ in B, we have $\varphi^{-1}(\mathfrak{q} \cap A) = \mathfrak{q}$.]

3. Let K be a field and let L be a totally inseparable extension of K such that $[L : K] = q$. Let A be a DVR whose field of fractions is K. Let v the discrete valuation on K^* such that A is $\{0\}$ union the set of all $k \in K^*$ with $v(k) \geq 0$. Let n be the smallest positive integer such that there is an $\ell \in L^*$ with $n = v(\ell^q) > 0$. Define $w : L^* \longrightarrow \mathbb{Z}$ by

$$w(\ell) = \frac{v(\ell^q)}{n}.$$

Show that w is a discrete valuation on L^*.

4. Let K be a field and let L be a totally inseparable extension of K such that $[L : K]$ is a finite.

(1) Let A be a DVR whose field of fractions is A and let B be the integral closure of A in L. Show that B is a DVR.

(2) Let A be a Dedekind domain whose field of fractions is A and let B be the integral closure of A in L. Show that B is a Dedekind domain.

[Hint: Use previous problems!]

5. Let A be a DVR with maximal ideal $\mathcal{P}$, field of fractions K. Let L be a finite separable extension of K and let B be a ring in L that is integral over A and has field of fractions L. Let $\mathcal{Q}$ be a prime in B for which $\mathcal{Q} \cap A = \mathcal{P}$, $\mathcal{Q}^e \supseteq B\mathcal{P}$ and $[B/\mathcal{Q} : A/\mathcal{P}] = f$. Show that $\dim_{A/\mathcal{P}}(B/\mathcal{Q}^e) \geq ef$ with equality if and only if $B_{\mathcal{Q}}\mathcal{Q}$ is principal or $e = 1$.

6. Let A be a DVR with maximal ideal $\mathcal{P}$, field of fractions K. Let L be a finite separable extension of K of degree n and let B be a ring in L that is integral over A and has field of fractions L. Suppose that $B\mathcal{P}$ factors as

$$B\mathcal{P} = \mathcal{Q}_1^{e_1} \cdots \mathcal{Q}_m^{e_m}.$$

Let f_i be the relative degree $[B/\mathcal{Q}_i : A/\mathcal{P}]$ of $\mathcal{Q}_i$ over $\mathcal{P}$. Show that $\sum_{i=1}^{m} e_i f_i = n$ if and only if B is Dedekind.

7. Let p be a prime number. Show that $\mathbb{Z}[i]p$ factors as

$$\begin{array}{rcl} \mathcal{Q}^2 & ; & \text{if } p = 2 \\ \mathcal{Q}_1\mathcal{Q}_2 & ; & \text{if } p \equiv 1 \pmod 4 \\ \mathcal{Q} & ; & \text{if } p \equiv 3 \pmod 4, \end{array}$$

where $\mathcal{Q}, \mathcal{Q}_1, \mathcal{Q}_2$ are primes of $\mathbb{Z}[i]$ and $\mathcal{Q}_1 \neq \mathcal{Q}_2$.