

Math 568 Problem Set #4 Due 10/13/14

1. Let R be a finite ring with $R \neq 0$. Show that R is zero-dimensional.
2. Let R be an integral domain, let \mathcal{M} be a maximal ideal of R , and let S be a multiplicative subset of R contained in $R \setminus \mathcal{M}$. Show that

$$R/\mathcal{M}^n \cong S^{-1}R/(S^{-1}R\mathcal{M}^n).$$

3. Let $\theta = \sqrt{3}$ and let $R = \mathbb{Z}[\theta]$.
 - (a) Write down a dual basis for the basis $1, \theta$ for $\mathbb{Q}(\theta)$ over \mathbb{Q} with respect to the bilinear form $(x, y) = \text{Tr}_{\mathbb{Q}(\theta)/\mathbb{Q}}(xy)$.
 - (b) Letting R^\dagger denote the \mathbb{Z} -module generated by the dual basis above, find the order of the abelian group R^\dagger/R .
4. Let K be a field. We define the resultant $\text{Res}(f, g)$ as follows. Write

$$f(x) = b \prod_{i=1}^m (x - \alpha_i) \quad , \quad g(x) = c \prod_{j=1}^n (x - \beta_j),$$

with $b, c \in K^*$ and β_i, γ_j in some algebraic closure of K . Then

$$\text{Res}(f, g) := b^n c^m \prod_{i=1}^m \prod_{j=1}^n (\alpha_i - \beta_j).$$

- (a) Let $h(x) = x^2 - 3$. Calculate $\text{Res}(h(x), h'(x))$.
 - (b) Let $t(x) = x^2 - x - 1$. Calculate $\text{Res}(t(x), t'(x))$.
5. Let f and g be as above and suppose that g is nonconstant. Show that

$$\text{Res}(f(x), g(x)) = b^n \prod_{i=1}^m g(\alpha_i).$$

6. Let f be as above and assume that its leading coefficient b is 1 and that the degree of f is at least 2. Show that

$$\text{Res}(f(x), f'(x)) = \prod_{\substack{1 \leq i, j \leq m \\ i \neq j}} (\alpha_i - \alpha_j).$$

7.
 - (a) Let $K \subseteq L$ be a separable field extension with $L = K(\theta)$ for an algebraic θ with minimal monic polynomial $f(x)$ over K . Show that for $a \in K$, we have $N_{L/K}(a - \theta) = f(a)$.
 - (b) Let ξ_p be a primitive p -th root of unity for a prime number p . Show that $N_{\mathbb{Q}(\xi_p)/\mathbb{Q}}(1 - \xi_p) = p$.