Math 568 Problem Set #3 Due 10/6/14

1. (p.14, Ex. 2) Let R be a Noetherian integral domain with field of fractions K and let M be an R-submodule of a finite dimensional R-vector space. Prove that

$$M = \bigcap_{\mathcal{P} \text{ maximal}} R_{\mathcal{P}} M.$$

2. Let K be a field and let F be a polynomial of positive degree with coefficients in K, which factors as

$$F(X) = \prod_{i=1}^{n} (f_i(X))^{e_i}$$

where e_i are positive integers, the f_i have positive degree, and $K[X]f_i + K[X]f_j = 1$ for $i \neq j$.
(a) Show that

$$K[X]/(F(X)) \cong \sum_{i=1}^{n} K[X]/(f_i(X)^{e_i}).$$

(b) Show that $K[X]/(F(X))$ is Noetherian and has dimension 0.

3. Let $A = \mathbb{Z}[\frac{1+\sqrt{-7}}{2}]$, considered as a subset of the complex numbers $\mathbb{C}$. Let $\mathrm{N} : \mathbb{C} \longrightarrow \mathbb{R}$ be the usual norm map given by $\mathrm{N}(a+bi) = a^2+b^2$. Let $m \in A$ be nonzero. Show that for any $n \in A$, there is a $q \in A$ and such that $\mathrm{N}(n - mq) < \mathrm{N}(m)$. Conclude that A is a principal ideal domain.

4. Let $A = \mathbb{Z}[\frac{1+\sqrt{-11}}{2}]$, considered as a subset of the complex numbers $\mathbb{C}$. Let $\mathrm{N} : \mathbb{C} \longrightarrow \mathbb{R}$ be the usual norm map given by $\mathrm{N}(a+bi) = a^2+b^2$. Let $m \in A$ be nonzero. True or false and explain: for any $n \in A$, there is a $q \in A$ and such that $\mathrm{N}(n - mq) < \mathrm{N}(m)$.

5. Let $A = \mathbb{Z}[\frac{1+\sqrt{-15}}{2}]$, considered as a subset of the complex numbers $\mathbb{C}$. Let $\mathrm{N} : \mathbb{C} \longrightarrow \mathbb{R}$ be the usual norm map given by $\mathrm{N}(a+bi) = a^2+b^2$. Let $m \in A$ be nonzero. True or false and explain: for any $n \in A$, there is a $q \in A$ and such that $\mathrm{N}(n - mq) < \mathrm{N}(m)$.

6. Let I be a nonzero ideal in a Noetherian integral R domain of dimension 1. Show that I factors as a product of primes if and only if $R_{\mathcal{M}} I$ is a power of $R_{\mathcal{M}} \mathcal{M}$ for every maximal ideal $\mathcal{M}$ of R. You may find Lemma 3.18 from the book to be helpful.

7. Let d be a square-free integer congruent to 1 modulo 4 that is not congruent to 1 modulo 8. Let $R = \mathbb{Z}[\sqrt{d}]$. Let I be the ideal generated by 2 in R, let J be the ideal generated by $1 - \sqrt{d}$ in R, and let $\mathcal{P} = I + J$. Show that

(1) $R/\mathcal{P}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and that $\mathcal{P}$ is therefore actually a prime;
(2) $\mathcal{P}^2 \subseteq I$;
(3) $|R_\mathcal{P}/(R_\mathcal{P}I)| = |R_\mathcal{P}/(R_\mathcal{P}J)|$ but that $R_\mathcal{P}I \neq R_\mathcal{P}J$;
(4) $R_\mathcal{P}J$ is not a power of $R_\mathcal{P}\mathcal{P}$ and J therefore cannot be factored as a product of prime ideals.

8. Let R be an integral integral domain and let M and N be fractional ideals of R. Do the following:

(1) Show that if M is invertible, then $(R : M)(R : N) = (R : MN)$.
(2) Let R and $\mathcal{P}$ be as in problem #7. Show that $(R : \mathcal{P})(R : \mathcal{P}) \neq (R : \mathcal{P}^2)$.

9. Let R be a Noetherian domain with the property that every *prime* ideal is principal. Show that every ideal of R is principal. [Hint: You may want to begin by showing that R is Dedekind]

10. We will say that a ring R is unique factorization domain (UFD) if R is an integral domain and if

- every nonunit $a \in R$ can be written as $\prod_{i=1}^n \pi_i^{e_i}$, where $e_i \in \mathbb{Z}^+$ and $R\pi_i$ is a prime ideal in R; and
- given two factorizations
$$a = \prod_{i=1}^n \pi_i^{e_i} = \prod_{i=1}^m \gamma_i^{f_i},$$
where $e_i, f_i \in \mathbb{Z}^+$ and $R\pi_i$, $R\gamma_i$ are prime ideals in R, we must have $m = n$ and a reordering σ of $1, \ldots, n$ such that $R\pi_i = R\gamma_{\sigma(i)}$ and $e_i = f_{\sigma(i)}$.

Since a principal ideal domain is a Dedekind domain or a field, it follows from unique factorization for ideals in a Dedekind domain that a principal ideal domain is a UFD. Show the partial converse: any Noetherian UFD of dimension 1 is a principal ideal domain.