Math 568 Problem Set #2 Due 9/22/14

1. (a) Let $\phi : A \longrightarrow B$ be a mapping of rings. Show that for any prime ideal $\mathcal{P}$ in B, the ideal $\phi^{-1}(\mathcal{P})$ is a prime ideal in A.

(b) Give an example of a surjective ring homomorphism $\phi : A \longrightarrow B$ for which there is a prime ideal $\mathcal{P}$ of A such that $\phi(\mathcal{P})$ is *not* a prime ideal.

2. (a) Give an example of a mapping of rings $\phi : A \longrightarrow B$ for which there is an ideal I of A such that $\phi(I)$ is not an ideal.

(b) Let $\phi : A \longrightarrow B$ be a surjective mapping of rings. Show that for any ideal I of A, the set $\phi(I)$ forms an ideal in B.

(c) Let $\phi : A \longrightarrow B$ be any mapping of rings. Show that for any ideal J of B, the set $\phi^{-1}(J)$ forms an ideal in A.

3. Let $A \subset B$ where A and B are domains and let K be the field of fractions of B. Show that if B is integrally closed over A in K, then B is integrally closed over itself in K.

4. (Ex. 4, p.3) Show that if S is a multiplicative set not containing 0 in a Noetherian integral domain R, then $S^{-1}R$ is also a Noetherian integral domain.

5. The definition of a Noetherian R-module for a ring R is very similar to that of a Noetherian ring. We say that M is a Noetherian R-module if it satisfies the ascending module property, which says that given any ascending chain R-submodules of R as below

$$M_0 \subseteq M_1 \subseteq \cdots \subseteq M_j \subseteq \ldots$$

there is some N such that $M_n = M_{n+1}$ for all $n \geq N$. As with rings, this is equivalent to saying that all of the R-submodules of M are finitely generated.

Let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of R-modules. Show that M is a Noetherian R-module if and only if M' and M'' are Noetherian R-modules.

6. Let R be a Noetherian integral domain, let I be an ideal of R,and let $S \subset R$ be a nonempty multiplicative set with $0 \notin S$. Let φ be the usual map from R to $S^{-1}R$. Show that if $S \cap I$ is empty, then $R_S\phi(I)$ is not all of R_S.

7. Let R be a ring and let $\phi : R \longrightarrow R/I$ be the natural quotient map.

(a) Show that the map

$$\phi^{-1} : J \longrightarrow \phi^{-1}(J)$$

from ideals in R/I to ideals in R gives a bijection between the set of ideals in R/I and the set set of ideals in R that contain I.

(b) Show that the map ϕ^{-1} from prime ideals in R/I to prime ideals in R gives a bijection between the set of prime ideals in R/I and the set set of prime ideals in R that contain I.

8. Find a ring R and an ideal I for which there is an element $c \in I^2$ that cannot be written as ab where $a, b \in I$.

9. (p. 6, Ex.3) Show that if $\{R_i\}$ is a family of integrally closed subrings of a field K, then the intersection

$$\bigcap_i R_i$$

is also integrally closed.