Math 568 Tom Tucker
NOTES FROM CLASS 11/19

Theorem 31.1. *Let $I \subset \mathcal{O}_L$ be any fractional ideal of $\mathcal{O}_L$. Then there exists an ideal $J \subset \mathcal{O}_L$ in the same ideal class as I such that*

$$|\operatorname{N}_{L/\mathbb{Q}}(J))| \leq \frac{n!}{n^n}\left(\frac{4}{\pi}\right)^s \sqrt{\Delta(\mathcal{O}_L/\mathbb{Z})}.$$

Corollary 31.2. $|\operatorname{Cl}(\mathcal{O}_L)|$ *is finite.*

Proof. There are finitely many ideals in $\mathcal{O}_L$ of bounded norm. □

Example 31.3. $|\operatorname{Cl}(\mathbb{Z}[\sqrt{-13}])| = 2$. We check the Minkowski bound and find it to be smaller than 5:

$$(1/2)(4/\pi)2\sqrt{13} < 5.$$

So we only need to check at 2 and 3. Let's see what happens at 2:

$$x^2 + 13 \equiv x^2 + 1 \pmod 2 \equiv (x+1)^2 \pmod 2.$$

Let $\mathcal{P} = (\sqrt{-13}+1, 2)$. I claim it isn't principal. To start with, we see that 2 is irreducible in $\mathbb{Z}[\sqrt{-13}]$. If it weren't irreducible, we could write $xy = 2$ with x, y not units. Let N denote $\operatorname{N}_{\mathbb{Q}(\sqrt{-13})/\mathbb{Q}}$. This would mean that $\operatorname{N}(x)\operatorname{N}(y) = \operatorname{N}(2) = 4$. Since x and y are not units, we cannot have $\operatorname{N}(x)$ or $\operatorname{N}(y)$ equal to 2. But there are no a and b with $a^2 + 13b^2 = 2$, so this is impossible. Thus, the only possible generator for the ideal $\mathcal{P}$ is 2. But $2 \nmid (\sqrt{-13}+1)$, so 2 cannot generate this ideal. In fact, we already know that $\mathcal{P}^2 = 2$, so 2 cannot generate this ideal. Hence $[\mathcal{P}] \neq 1$, where $[\mathcal{P}]$ is the ideal class of $\mathcal{P}$. Since $\mathcal{P}^2 = (2)$, we know that $[\mathcal{P}]^2 = 1$.

Let's look over 3 now:

$$x^2 + 13 \equiv x^2 + 1 \pmod 3$$

which is irreducible, so $3\mathbb{Z}[\sqrt{-13}]$ is the only prime lying over 3. Thus, $\operatorname{Cl}(\mathbb{Z}[\sqrt{-13}])$ is generated by $[\mathcal{P}]$ above and thus has order 2.

Example 31.4. $|\operatorname{Cl}(\mathbb{Z}[\frac{\sqrt{13}+1}{2}])| = 1$. Plugging into Minkowski, we get

$$(1/2)\sqrt{13} < 2$$

so we must have class number 1.