Math 568 Tom Tucker
NOTES FROM CLASS 10/27/14

From last time, we have the following.

Corollary 13.1. *For each prime $\mathcal{P}$ of A, we have $A_{\mathcal{P}}\Delta(B'/A) = \Delta(S^{-1}B'/A_{\mathcal{P}})$, where $S = A \setminus \mathcal{P}$.*

Thus, we can calculate the discriminant locally.

The trace also behaves well with respect to reduction. Whenever we have a finite integral extension of a field, we can define a trace. We'll apply that with the field $k = A/\mathcal{P}$ for a maximal ideal $\mathcal{P}$ of A. Since this computation is local, we will work over $A_{\mathcal{P}}$ (which is a DVR). The following applies whenever B' is a free A-module.

Lemma 13.2. *Let A and B' be as usual. Suppose that B' is a free A-module. Let $\mathcal{P}$ be a nonzero prime of A, let $k = A/\mathcal{P}A$, let $C = B/\mathcal{P}B$, and let $\phi : B \longrightarrow C$ be the usual reduction map. Then for any $y \in B'$, we have $\phi(T_{L/K}(y)) = T_{C/k}(\phi(y))$.*

Proof. Let $\{w_1, \ldots, w_n\}$ be basis for B' as an A-module. Then, letting $\bar{w}_i$ denote $\phi(w_i)$, we see that $\{\bar{w}_1, \ldots, \bar{w}_n\}$ must generate C as a k-vector space and thus must be a basis for C as a k-vector space.

Now, we are essentially done, since we can define the trace of any $y \in B'$ with respect to this basis. We have

$$yw_i = \sum_{j=1}^{n} m_{ij} w_j$$

with $m_{ij} \in A$, and

$$\phi(y)\bar{w}_i = \sum_{j=1}^{n} \phi(m_{ij})\bar{w}_j.$$

Hence,

$$\phi(\mathrm{T}_{L/K}(y)) = \sum_{i=1}^{n} \phi(m_{ii}) = \mathrm{T}_{C/k}(\phi(y)).$$

□

We need one quick lemma from linear algebra.

Lemma 13.3. *Let V be a vector space. Let $\phi : V \longrightarrow V$ be a linear map. Suppose that $\phi^k = 0$ for some $k \geq 1$. Then the trace of ϕ is zero.*

Proof. This is on your HW. □

When B is the integral closure of A in L, and $\mathcal{P}$ is maximal in A, we can write

$$\mathcal{P}B = \mathcal{Q}_1^{e_1} \cdots \mathcal{Q}_m^{e_m}.$$

If $e_i > 1$ for some i, then we say that $\mathcal{P}$ *ramifies* in B. When $B = A[\alpha]$, we know that $\mathcal{P}$ ramifies in B if and only if $\Delta(B/A) \subseteq \mathcal{P}$. That is true more generally.

We make one more assumption today: we assume that every residue field $A/\mathcal{P}$, for $\mathcal{P}$ a nonzero prime, is *perfect*. That is, we assume that it has no inseparable extensions.

Theorem 13.4. *Let B be the integral closure of A in L and let $\mathcal{P}$ be maximal in A. Then $\mathcal{P}$ ramifies in B if and only if $\Delta(B/A) \subseteq \mathcal{P}$.*

Proof. It will suffice to prove this locally, that is to say, it will suffice to replace A with $A_{\mathcal{P}}$ and B with $S^{-1}B$ where $S = A \setminus \mathcal{P}$. Thus, we may assume that A is a DVR and that B is its integral closure in L.

Let $w_1, \ldots, w_n$ be a basis for B over A. From the Lemma above we have $T_{L/K}(w_i w_j) = T_{C/k}(\bar{w}_i \bar{w}_j)$, so the matrix $M = [\mathrm{T}_{C/k}(\bar{w}_i \bar{w}_j)]$ represents the form $(x, y) = T_{C/k}(xy)$ on C/k. We have $\det M = 0$ (in $A/\mathcal{P}$) if and only if the form is degenerate, i.e. if and only if there is some nonzero x such that $T_{C/k}(xy) = 0$ for all $y \in C$. Since $\det M$ is simly $\Delta(A/B)$, we we need only show then that the form is degenerate exactly when we have $e_i > 1$ for some $\mathcal{Q}_i$ in the factorization

$$\mathcal{P}B = \mathcal{Q}_1^{e_1} \ldots \mathcal{Q}_m^{e_m}$$

Let us now decompose C/k as ring, we have

$$C \cong B/\mathcal{P}B \cong \bigoplus_{i=1}^{m} B/\mathcal{Q}_i^{e_i}$$

where

$$\mathcal{P}B = \mathcal{Q}_1^{e_1} \cdots \mathcal{Q}_m^{e_m}.$$

If $e_i > 1$, then any element $z \in C$ such that $z = 0$ in every coordinate but i and has i-th coordinate in $\mathcal{Q}_i$, has the property that $z^{e_i} = 0$. Thus, by your homework we must have $T_{C/k}(z) = 0$ for all such z. Since the set of such elements forms a C-ideal this means that we have $T_{C/k}(zy) = 0$ for all $y \in C$. Hence the pairing

$$(x, y) = T_{C/k}(xy)$$

on C is degenerate.

If $e_i = 1$ for every i, then

$$C \cong B/B\mathcal{Q}_1 \oplus \cdots \oplus B/S^{-1}B\mathcal{Q}_m$$

and $B/\mathcal{Q}_i$ is separable over k for each i. The trace form $(x, y) = \mathrm{T}_{C/k}(xy)$ decomposes into a sum of forms

$$(a, b) = \mathrm{T}_{(B/\mathcal{Q}_i)/k}(ab),$$

each of which is nondegenerate, so (x, y) is nondegenerate, so

$$\det[\mathrm{T}_{L/K}(w_i w_j)] \notin \mathcal{P},$$

and we are done.

□

Here is a simple and easy to prove fact comparing the discriminants of different subrings B and B' of L

Corollary 13.5. *Suppose that A is a DVR. Let $B' \subset B$ with B' and B as usual. Then*

$$\Delta(B/A)(\Delta(B'/A))^{-1} = \mathcal{P}^{2r}$$

with $r > 0$ unless $B = B'$.

Proof. This is on your homework. □

Corollary 13.6. *Let $\mathcal{P}$ be a prime of A. If $\Delta(B'/A)$ is not contained in $\mathcal{P}^2$, then $S^{-1}B'$ is integrally closed where $S = A \setminus \mathcal{P}$.*

Proof. Let B be the integral closure of A in the field of fractions of B. Then $\Delta(S^{-1}B/A_{\mathcal{P}})$ must equal $\Delta(S^{-1}B'/A_{\mathcal{P}})$ by the Corollary above since $\Delta(S^{-1}B'/A)$ does not contain $A_{\mathcal{P}}\mathcal{P}^2$. Thus, $S^{-1}B' = S^{-1}B$, so $S^{-1}B'$ is integrally closed. □