Math 568 Tom Tucker
NOTES FROM CLASS 10/22/14

For the rest of class, A is Dedekind with field of fractions K, the field L is a finite separable extension of K of degree n, and B' is a subring of L that is integral over A. We will also assume that for every maximal ideal $\mathcal{P}$ of A.

We'll begin with a definition that works when B' is a free A-module, i.e. when B' is isomorphic as an A-module to A^n, where $n = [L : K]$. In this case, we choose a basis $w_1, \ldots, w_n$ for B' over A and we let M be the matrix $[m_{ij}]$ where $m_{ij} = \mathrm{T}_{L/K}(w_i w_j)$. Then we define

$$\Delta(B') = \det M. \tag{1}$$

How do we know that this agrees with our earlier definition in the case $B' = A[\alpha]$? In fact, it more or less follows from some earlier work we did. Recall that in this case, we can choose the basis $1, \alpha, \ldots, \alpha^{n-1}$, so that $[m_{ij}] = [\mathrm{T}_{L/K}(\alpha^{i+j-2})]$, which we recall is equal to

$$\sum_{\ell=1}^{n} \alpha_\ell^{i+j-2}.$$

As we saw earlier, letting N be the van der Monde matrix

$$\begin{pmatrix} 1 & \cdots & 1 \\ \alpha_1 & \cdots & \alpha_n \\ \cdots & \cdots & \cdots \\ \alpha_1^{n-1} & \cdots & \alpha_n^{n-1} \end{pmatrix},$$

we have $NN^t = M$, so

$$\det M = (\det N)^2 = \prod_{i<j} (\alpha_i - \alpha_j)^2,$$

which is the same as $\Delta(\alpha)$, so our definitions agree.

Not all B' will be free A-modules, however, so we have the more general definition below.

Definition 12.1. With notation as above $\Delta(B'/A)$ is defined to be ideal generated by the determinants of all matrices $M = [\mathrm{T}_{L/K}(w_i w_j)]$ as $w_1, \ldots, w_n$ range over all bases for L consisting of elements contained in B'.

To see that this definition agrees with our earlier definition for when B' is a free A-module. The following lemma will be on your homework.

Lemma 12.2. *Suppose that $B' = Av_1 + \ldots Av_n$ where $\{v_1, \ldots, v_n\}$ is a basis for L over K. Then for other basis $W = \{w_1, \ldots, w_n\}$ for L*

over K with $W \subseteq B$, we have

$$\det[T_{L/K}(w_i w_j)]_{i,j} = a^2 \det[T_{L/K}(v_i v_j)]_{i,j}$$

for some $a \in A$.

Proof. This will be on the homework. □

Example 12.3. The reason that we need to talk about the discriminant relative to A is that B' could be defined over two different Dedekind domains. For example, we could take $B' = \mathbb{Z}[\sqrt{3}, \sqrt{7}]$ which is an extension of $\mathbb{Z}$ as well as of $\mathbb{Z}[\sqrt{3}]$ and $\mathbb{Z}[\sqrt{7}]$. The various discriminants $\Delta(B'/\mathbb{Z})$, $\Delta(B'/\mathbb{Z}[\sqrt{3}])$, and $\Delta(B'/\mathbb{Z}[\sqrt{7}])$ may all be different.

One nice fact about discriminants is that they can be computed locally. We have the following.

Proposition 12.4. *With notation as throughout lecture, let S be a multiplicative subset of A not containing 0. Then*

$$S^{-1}A\Delta(B'/A) = \Delta(S^{-1}B'/S^{-1}A).$$

Proof. Since any basis with elements in B' is also in $S^{-1}B'$, it is obvious that

$$S^{-1}A\Delta(B'/A) \subseteq \Delta(S^{-1}B'/S^{-1}A).$$

Similarly, given a basis $v_1, \ldots, v_n$ for L/K contained in $S^{-1}B'$, see that the basis $w_1, \ldots, w_n$ where $w_i = sv_i$ is contained in B' for some $s \in S$. Now

$$\det(T_{L/K}(w_i w_j)) = s^{2n} \det(T_{L/K}(v_i v_j)),$$

so $S^{-1}A\Delta(B'/A) \supseteq \Delta(S^{-1}B'/S^{-1}A)$. □

We know that $\Delta(B'/A)$ is an ideal I. If $I = \prod_{i=1}^{m} \mathcal{P}_i^{e_i}$, then $A_{\mathcal{P}_i} I = \mathcal{P}_i^{e_i}$, so to figure out what $\Delta(B'/A)$ is, all we have to do is figure out what $\Delta(S^{-1}B'/S^{-1}A)$ is for $S = A \setminus \mathcal{P}$.