Math 568 Problem Set #1 Due 9/15/14

1. For an element $u + vi$ of $\mathbb{Z}[i]$, define $N(u + vi) := u^2 + v^2$.
(a) Show that for any $c, d \in \mathbb{Z}[i]$, we have $N(c)N(d) = N(cd)$.
(b) Show that for any elements $a, b \in \mathbb{Z}[i]$, it is possible to write

$$a = qb + r,$$

where $q, r \in \mathbb{Z}[i]$ and $N(r) < N(b)$.
(c) Conclude that $\mathbb{Z}[i]$ is a principal ideal domain.

2. Use the Euclidean algorithm to show that $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ is a principal ideal domain.

3. Answer each of the following yes or no and explain your answer.

(a) Is $11\sqrt{7}$ integral over $\mathbb{Z}$?

(b) Is $\frac{1+\sqrt{3}}{2}$ integral over $\mathbb{Z}$?

(c) Is $\frac{1+\sqrt{5}}{2}$ integral over $\mathbb{Z}$?

(d) Is $\mathbb{Z}[\sqrt{-19}]$ integrally closed in $\mathbb{Q}[\sqrt{-19}]$?

4. Show that ± 1 are the only units in the ring $\mathbb{Z}[\frac{1+\sqrt{-19}}{2}]$.

5. It turns out that $\mathbb{Z}[\frac{1+\sqrt{-19}}{2}]$ is a unique factorization domain (we will prove this later). Given this fact, find *all* integer pairs (x, y) such that $x^2 + 19 = y^3$ and justify your answers with a proof.

6. (a) Suppose that α is integral of degree 2 over $\mathbb{Z}$ and that there is a basis m_1, m_2 for $\mathbb{Z}[\alpha]$ over $\mathbb{Z}$ such that

$$\alpha m_1 = m_2$$
$$\alpha m_2 = -m_1 + m_2.$$

Find the quadratic integral equation satisfied by α.

(b) Suppose that β is integral of degree 2 over $\mathbb{Z}$ and that there is a basis m_1, m_2 for $\mathbb{Z}[\beta]$ over $\mathbb{Z}$ such that

$$\beta m_1 = m_2$$
$$\beta m_2 = 5m_1.$$

Find the quadratic integral equation satisfied by β.

7. Let T be an $n \times n$ matrix with coefficients in a ring A. Show that there is a matrix U for which $UT = (\det T)I$.

8. Let A, B, and C be rings with $A \subset B \subset C$. Show that if B is integral over A and C is integral over B, then C is integral over A.

9. (Ex. 4, p. 7) Let d be a squarefree integer. Show that the integral closure of $\mathbb{Z}$ in $\mathbb{Q}[\sqrt{d}]$ is

$$\mathbb{Z}[\sqrt{d}] \qquad \text{if} \quad d \equiv 2, 3 \pmod 4,$$

and

$$\mathbb{Z}\left[\frac{1+\sqrt{d}}{2}\right] \qquad \text{if} \quad d \equiv 1 \pmod 4.$$