

Math 531 Problem Set #4 Due 9/29/04

1. Let I be a nonzero ideal in a Noetherian integral R domain of dimension 1. Show that I factors as a product of primes if and only if $R_{\mathcal{M}}I$ is a power of $R_{\mathcal{M}}\mathcal{M}$ for every maximal ideal \mathcal{M} of R . You may find Lemma 3.18 from the book to be helpful.

2. Let d be a square-free integer congruent to 1 modulo 4 that is not congruent to 1 modulo 8. Let $R = \mathbb{Z}[\sqrt{d}]$. Let I be the ideal generated by 2 in R , let J be the ideal generated by $1 - \sqrt{d}$ in R , and let $\mathcal{P} = I + J$. Show that

- (a) R/\mathcal{P} is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and that \mathcal{P} is therefore actually a prime;
- (b) $\mathcal{P}^2 \subseteq I$;
- (c) $R_{\mathcal{P}}/(R_{\mathcal{P}}I) \cong R_{\mathcal{P}}/(R_{\mathcal{P}}J)$ (as rings) but that $R_{\mathcal{P}}I \neq R_{\mathcal{P}}J$;
- (d) $R_{\mathcal{P}}J$ is not a power of $R_{\mathcal{P}}\mathcal{P}$ and J therefore cannot be factored as a product of prime ideals.

3. Let R be an integral integral domain and let M and N be fractional ideals of R . Do the following:

- (a) Show that if M is invertible, then $(R : M)(R : N) = (R : MN)$.
- (b) Let R and \mathcal{P} be as in problem #2. Show that $(R : \mathcal{P})(R : \mathcal{P}) \neq (R : \mathcal{P}^2)$.

4. Let R be a Noetherian domain with the property that every *prime* ideal is principal. Show that every ideal of R is principal. [Hint: You may want to begin by showing that R is Dedekind]

5. We will say that a ring R is unique factorization domain (UFD) if R is an integral domain and if

- every nonunit $a \in R$ can be written as $\prod_{i=1}^n \pi_i^{e_i}$, where $e_i \in \mathbb{Z}^+$ and $R\pi_i$ is a prime ideal in R ; and
- given two factorizations

$$a = \prod_{i=1}^n \pi_i^{e_i} = \prod_{i=1}^m \gamma_i^{f_i},$$

where $e_i, f_i \in \mathbb{Z}^+$ and $R\pi_i, R\gamma_i$ are prime ideals in R , we must have $m = n$ and a reordering σ of $1, \dots, n$ such that $R\pi_i = R\gamma_{\sigma(i)}$ and $e_i = f_{\sigma(i)}$.

Since a principal ideal domain is a Dedekind domain or a field, it follows from unique factorization for ideals in a Dedekind domain that a principal ideal domain is a UFD. Show the partial converse: any Noetherian UFD of dimension 1 is a principal ideal domain. [You may use Problem # 4]

6. Let R be a finite ring with $R \neq 0$. Show that R is zero-dimensional.