Problem Set #13 for May 3, 2006

1. Prove that if D is an effective divisor (i.e., $D \geq 0$), on a curve C, then $\dim D = \deg D + 1$ if and only if $D = 0$ or the genus of C is zero.

2. The purpose of this exercise is to show that for any curve nonsingular projective C, there is a map from $\varphi : C \longrightarrow \mathbb{P}^2$ that is a birational isomorphism.

(a) Let p be the characteristic of the base field k. Prove that there is a map $f : C \longrightarrow \mathbb{P}^1$ such that the degree of f is prime to p.

(b) Let f be your map from (a). Show that there is an element u of $K(C)$ such that $K(C) = k(f, u)$. [Hint: use the primitive element theorem for finite, separable extensions.]

(c) Show that the map from C to $\mathbb{P}^2$ induced by $[f : u : 1]$ is birational. [Hint: it suffices to show that for some affine subset $\mathcal{U}$ of C the map to $\mathbb{A}^2$ given by (f, u) is birational. Choose an affine subset on which f and u are regular.]

3. Let C be the nonsingular projective model for the affine curve $y^2 = f(x)$ where f is a polynomial of degree greater than two without repeated roots. Assume that the base field has characteristic not equal to 2. Calculate the genus of C using the Riemann-Hurwitz formula.

4. Let C be the projective nonsingular model for the affine curve given by $y^3 = x^3 - x$. Assume the characteristic of the base field isn't equal to 2 or 3.

(a) Show that the rational map given by projection onto the x-axis ramifies only at $(0, 0)$, $(0, 1)$, and $(0, -1)$. Denote these points as P_1, P_2, and P_3; for each i, let Q_i denote the image of P_i in $\mathbb{P}^1$.

(b) For z in $K(C)$, let $\mathrm{Tr}(z)$ denote the trace of the map induced by multiplication by z. Show that for any element of the $z = f(x) + g(x)y + h(x)y^2$, we have $\mathrm{Tr}(z) = 3f(x)$.

(c) Show that for each i, we have $v_{Q_i}(\mathrm{Tr}(z)) \geq 0$ whenever $v_{P_i}(z) \geq -2$.

(d) Show that for any point P' that is not one of the P_i, there is an element z such that $v_{P'}(z) = -1$ and $v_{Q'}(\mathrm{Tr}(z)) = -1$, where Q' is the image of P'.

(e) Let ω be the Weil differential on $\mathcal{A}_{k(x)}$ that vanishes at $\mathcal{A}_{k(x)}(-2\infty)$ (i.e. the differential coming from the residue map for dx). Show that the differential $\omega \circ \mathrm{Tr}$ (which sends $\mathcal{A}_{K(C)}$ to the base field k) vanishes on $\mathcal{A}_{K(C)}(D)$, where D is the divisor

$$D = 2(P_1 + P_2 + P_3) - 2(R_1 + R_2 + R_3)$$

and R_1, R_2, and R_3 are the points on C that have ∞ as their image when we project onto the x-axis.