Problem Set #12 for April 26, 2006

We will adopt the following notation: for a divisor $D = \sum_{P \in C} n_P P$, we let $v_P(D)$ denote the integer n_P. We let $\operatorname{Supp} D$ denote the set of all P for which $v_P(D) \neq 0$.

1. A linear system $\mathcal{L}(D)$ on a curve C is said to be *basepoint-free* if for any $P \in C$ there is a rational function $f \in \mathcal{L}(D)$ such that $v_P(f) = -v_P(D)$.
(a) Give an example of a linear system that is not basepoint-free.
(b) Show that $\mathcal{L}(D)$ is basepoint-free if and only if for any $P \in C$, we have

$$\dim(D - P) = (\dim D) - 1.$$

(c) Use (b) to deduce that for any curve C there is a constant M such that for all D with $\deg D > M$, the linear system $\mathcal{L}(D)$ is basepoint-free.

2. Let P and Q be distinct points in C. We say that a linear system $\mathcal{L}(D)$ separates P and Q if there is a rational function $f \in \mathcal{L}(D)$ such that $f(P) \neq f(Q)$.
(a) Take two distinct points P and Q, neither of which is in $\operatorname{Supp} D$. Show that $\mathcal{L}(D)$ separates P and Q whenever

$$\dim(D - P - Q) = \dim D - 2.$$

(b) Use (a) to deduce that for any curve C there is a constant M such that for all D with $\deg D > M$ and any two points P and Q outside of $\operatorname{Supp} D$, the system $\mathcal{L}(D)$ separates P and Q.

3. Let P be any point on a curve C. Show that for some n there is an injective map $\varphi : C \longrightarrow \mathbb{P}^n$ such that P is the only point in C whose image is in the hyperplane $x_n = 0$. [Hint: Take a large multiple mP of P and consider the linear system $\mathcal{L}(mP)$. A basis $f_0, \dots, f_{n-1}, 1$ for rational functions in $\mathcal{L}(mP)$ give a map from C to $\mathbb{P}^n$ for some n – think of it as $[f_0(P) : \cdots : f_{n-1}(P) : 1]$. Show that if m is large enough, then P is the only point that is sent into the hyperplane $x_n = 0$, using exercise 1. Then use exercise 2 to show that if m is large enough, then φ must be injective. On next week's HW, we'll show that φ is actually an isomorphism onto its image, which will prove that $C \setminus \{P\}$ is actually affine.]

4. We will say that a point P on C is a ramification point of the nonconstant rational function $f \in K(C)$ when one of the following conditions holds
(a) $v_P(f) \leq -2$;
(b) $v_P(f - f(P)) \geq 2$.

Note that the first condition simply means that f has a pole or order greater than 1 at P and that in the second $f(P)$ is simply an element of k so $f - f(P)$ is an element of the function field of C. The quantity e_P, which is defined to be $v_P(f)$ when P has a pole at P and $v_p(f - f(P))$ when f doesn't have a pole at P, is called the ramification index of f at P.
(a) Show that for any $\beta \in k$, we have

$$\#\{P \in C \mid f(P) = \alpha\} = [K(C) : k(f)]$$

unless one of the P such that $f(P) = \alpha$ is a ramification point for f.

(b) Show that if $C = \mathbb{P}^1$ and the rational function f is simply a polynomial in $k[x]$, then for any $\alpha \in k$, the corresponding point P is a ramification point for f if and only if $f'(\alpha) = 0$.

(c) With the same conditions as (b), suppose additionally that the characteristic of k is 0. Show that

$$\sum_{P \in \mathbb{P}^1} (e_P - 1) = 2 \deg f - 2.$$

(d) Give an example of a rational function f on a curve C such that *every point* on C is a ramification point of f. [This only works in characteristic $p > 0$]

[Note: this definition of ramification point is not standard. You may see it defined differently elsewhere]

5. The purpose of this exercise is to use basic facts about linear systems to give an upper bound on the genus of curves of a particular form. Let C be the projective curve defined by

$$zy^m - f(x, z) = 0$$

where $f(x, z)$ has degree $m + 1$ and factors into distinct linear factors and z does not divide $f(x, z)$.

(a) Show that C is nonsingular.

(b) Show that x/z has a pole of order m at $[0 : 1 : 0]$ and that y/z has a pole of order $m + 1$ at $[0 : 1 : 0]$.

(c) Let P denote the point $[0 : 1 : 0]$. Show that for any $N \geq m(m - 1)$, we have $\dim NP = \dim((N-1)P) + 1$. [Hint: it suffices to show you can write $N = sm + t(m+1)$ for positive integers s and t.]

(d) Use c to conclude that the genus of C is less than or equal to $\frac{m(m-1)}{2}$. [Just do a simple count here]