Problem Set #11 for April 19, 2006

Exercise 4 from Chapter I, Section 5 of Hartshorne. Plus

1. Let C be a nonsingular projective curve in $\mathbb{P}^n$ and let

$$f(X_0, \ldots, x_n) = \frac{g(X_0, \ldots, X_n)}{h(X_0, \ldots, X_n)}$$

be a rational function on C (here g and h have the same degree).
(a) Suppose that g is irreducible. Show that for any point P on C where $h(P) \neq 0$, we have

$$v_P(f) = i(C, Z(g); P),$$

where $i(C, Z; P)$ is the usual intersection multiplicity from Secion 7 of Hartshorne.
(b) Now, let C be a nonsingular projective curve in $\mathbb{P}^2$. Suppose that C doesn't contain the point $[0 : 0 : 1]$. Let $\pi : \mathbb{P}^2\ \{0\} \longrightarrow H$ be the usual projection map from $\mathbb{P}^2$ onto the hyperplane consistinig of all points of the form $[x : y : 0]$ and let $f : C \longrightarrow \mathbb{P}^1$ be the map this induces on C, thought of as a rational function where that vanishes when $x = 0$ and has poles when $y = 0$. Use Bézout's theorem to prove that $\sum_{v_P(f)\geq 0} v_P(f) = \deg C$.
(c) What happens when C contains $[0 : 0 : 1]$? Note that π still gives rise to a map on C in this case.

2. Let C be the nonsingular projective curve defined by $zy^2 = x^3 - z^2x$ in $\mathbb{P}^2$ (this is a projective curve that is birational to the curve defined by $y^2 = x^3 - x$, which we showed is not rational a few weeks ago). Let $P_1 = [0 : 1 : 0]$, let $P_2 = [2 : \sqrt{6} : 1]$, and let $P_3 = [2 : -\sqrt{6} : 1]$, all considered as points in $\mathbb{P}^2$. Find bases for the following linear systems on C.
(a) $\mathcal{L}(P_2)$.
(b) $\mathcal{L}(P_2 + P_3)$.
(c) $\mathcal{L}(P_1)$.
(d) $\mathcal{L}(2P_1)$.
(e) $\mathcal{L}(3P_1)$.

3. Let C be a projective curve and let $P_1, \ldots, P_n$ be any set of points in C. Show that there is an affine subset $\mathcal{U}$ of C that contains all of the P_i. [Try finding a hyperplane that doesn't contain any of the P_i.]

4.
(a) Let A be a Dedekind domain and let $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ be a set of maximal ideals. Let $B = \bigcap_{i=1}^n A_{\mathfrak{m}_i}$. Show that B has finitely many maximal ideals.
(b) Show that a Dedekind domain with finitely many maximal ideals must be a PID. [You may use the fact that any ideal in a Dedekind domain factors into a product of prime ideals]
(c) Use 3. along with (a) and (b) to show that given any distinct points $P_1, \ldots, P_n$ on a nonsingular projective curve C and any set of integers $e_1, \ldots, e_n$, there is an element t in the function field $K(C)$ such that $v_{P_i}(t) = e_i$ for $i = 1, \ldots, n$.

(Do four of the five)