Math 531 Problem Set #6 Due 10/13/04

1. Let A be a DVR with maximal ideal $\mathcal{P}$, field of fractions K. Let L be a finite separable extension of K and let B be a ring in L that is integral over A and has field of fractions L. Let $\mathcal{Q}$ be a prime in B for which $\mathcal{Q} \cap A = \mathcal{P}$ and such that $\mathcal{Q}^e \supseteq \mathcal{P}$ and $[B/\mathcal{Q} : A/\mathcal{P}] = f$. Show that $\dim_{A/\mathcal{P}}(B/\mathcal{Q}^e) \geq ef$ with equality if and only if $B_{\mathcal{Q}}\mathcal{Q}$ is principal or $e = 1$.

2. Let A be a DVR with maximal ideal $\mathcal{P}$, field of fractions K. Let L be a finite separable extension of K of degree n and let B be a ring in L that is integral over A and has field of fractions L. Suppose that $B\mathcal{P}$ factors as

$$B\mathcal{P} = \mathcal{Q}_1^{e_1} \cdots \mathcal{Q}_m^{e_m}.$$

Let f_i be the relative degree $[B/\mathcal{Q}_i : A/\mathcal{P}]$ of $\mathcal{Q}_i$ over $\mathcal{P}$. Show that $\sum_{i=1}^{m} e_i f_i = n$ if and only if B is Dedekind.

3. Let p be a prime number. Show that $\mathbb{Z}[i]p$ factors as

$$
\begin{aligned}
\mathcal{Q}^2 &\quad ; \quad \text{if } p = 2 \\
\mathcal{Q}_1\mathcal{Q}_2 &\quad ; \quad \text{if } p \equiv 1 \pmod 4 \\
\mathcal{Q} &\quad ; \quad \text{if } p \equiv 3 \pmod 4,
\end{aligned}
$$

where $\mathcal{Q}, \mathcal{Q}_1, \mathcal{Q}_2$ are primes of $\mathbb{Z}[i]$ and $\mathcal{Q}_1 \neq \mathcal{Q}_2$.

Also do the following problems from the book. You may assume that L is separable over K in all of them.

4. p. 29, Ex. 4

5. p. 30, Ex. 5

6. p. 32 Ex. 7