Math 531 Problem Set #5 Due 10/6/00

1. Let R be an integral domain, let $\mathcal{M}$ be a maximal ideal of R, and let S be a multiplicative subset of R contained in $R \setminus \mathcal{M}$. Show that

$$R/\mathcal{M}^n \cong S^{-1}R/(S^{-1}R\mathcal{M}^n).$$

2. Let $\theta = \sqrt{3}$ and let $R = \mathbb{Z}[\theta]$.
   (a) Write down a dual basis for the basis 1, θ for $\mathbb{Q}(\theta)$ over $\mathbb{Q}$ with respect to the bilinear form $(x, y) = \mathrm{T}_{\mathbb{Q}(\theta)/\mathbb{Q}}(xy)$.
   (b) Letting $R^\dagger$ denote the $\mathbb{Z}$-module generated by the dual basis above, find the order of the abelian group $R^\dagger/R$.
3. Let $\theta = \frac{1+\sqrt{5}}{2}$ and let $R = \mathbb{Z}[\theta]$.
   (a) Write down a dual basis for the basis 1, θ for $\mathbb{Q}(\theta)$ over $\mathbb{Q}$ with respect to the bilinear form $(x, y) = \mathrm{T}_{\mathbb{Q}(\theta)/\mathbb{Q}}(xy)$.
   (b) Letting $R^\dagger$ denote the $\mathbb{Z}$-module generated by the dual basis above, find the order of the abelian group $R^\dagger/R$.
4. Let K be a field. We define the resultant $\mathrm{Res}(f, g)$ as follows. Write

$$f(x) = b\prod_{i=1}^{m}(x - \alpha_i) \ \ , \ \ g(x) = c\prod_{j=1}^{n}(x - \beta_j),$$

with $b, c \in K^*$ and β_i, γ_j in some algebraic closure of K. Then

$$\mathrm{Res}(f, g) := b^n c^m \prod_{i=1}^{m}\prod_{j=1}^{n}(\alpha_i - \beta_j).$$

   (a) Let $h(x) = x^2 - 3$. Calculate $\mathrm{Res}(h(x), h'(x))$.
   (b) Let $t(x) = x^2 - x - 1$. Calculate $\mathrm{Res}(t(x), t'(x))$.
   (c) Compare your answers to 2(b) and 3(b).
5. Let f and g be as above and suppose that g is nonconstant. Show that

$$\mathrm{Res}(f(x), g(x)) = b^n \prod_{i=1}^{m} g(\alpha_i).$$

6. Let f be as above and assume that its leading coefficient b is 1 and that the degree of f is at least 2. Show that

$$\mathrm{Res}(f(x), f'(x)) = \prod_{\substack{1\le i,j\le m \\ i\ne j}} (\alpha_i - \alpha_j).$$

7.
   (a) Let $K \subseteq L$ be a separable field extension with $L = K(\theta)$ for an algebraic θ with minimal monic polynomial $f(x)$ over K. Show that for $a \in K$, we have $N_{L/K}(a - \theta) = f(a)$.
   (b) Let ξ_p be a primitive p-th root of unity for a prime number p. Show that $N_{\mathbb{Q}(\xi_p)/\mathbb{Q}}(1 - \xi_p) = p$.