Math 531 Problem Set #3 Due 9/22/04

1. The definition of a Noetherian R-module for a ring R is very similar to that of a Noetherian ring. We say that M is a Noetherian R-module if it satisfies the ascending module property, which says that given any ascending chain R-submodules of R as below

$$M_0 \subseteq M_1 \subseteq \cdots \subseteq M_j \subseteq \ldots$$

there is some N such that $M_n = M_{n+1}$ for all $n \geq N$. As with rings, this is equivalent to saying that all of the R-submodules of M are finitely generated.

Let M be a Noetherian R-module and let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of R-modules. Show that
(a) M' is a Noetherian R-module; and
(b) M'' is a Noetherian R-module.

2. Let R be a Noetherian integral domain, let I be an ideal of R,and let $S \subset R$ be a nonempty multiplicative set with $0 \notin S$. Let φ be the usual map from R to $S^{-1}R$. Show that if $S \cap I$ is empty, then $R_S\phi(I)$ is not all of R_S.

3. Let R be a ring and let $\phi : R \longrightarrow R/I$ be the natural quotient map.
(a) Show that the map

$$\phi^{-1} : J \longrightarrow \phi^{-1}(J)$$

from ideals in R/I to ideals in R gives a bijection between the set of ideals in R/I and the set set of ideals in R that contain I.

(b) Show that the map ϕ^{-1} from prime ideals in R/I to prime ideals in R gives a bijection between the set of prime ideals in R/I and the set set of prime ideals in R that contain I.

4. Find a ring R and an ideal I for which there is an element $c \in I^2$ that cannot be written as ab where $a, b \in I$.

5. (p. 6, Ex.3) Show that if $\{R_i\}$ is a family of integrally closed subrings of a field K, then the intersection

$$\bigcap_i R_i$$

is also integrally closed.

6. (p.14, Ex. 2) Let R be a Noetherian integral domain with field of fractions K and let M be an R-submodule of a finite dimensional

R-vector space. Prove that

$$M = \bigcap_{\mathcal{P} \text{ maximal}} R_{\mathcal{P}} M.$$

7. Let K be a field and let F be a polynomial of positive degree with coefficients in K, which factors as

$$F(X) = \prod_{i=1}^{n} (f_i(X))^{e_i}$$

where e_i are positive integers, the f_i have positive degree, and $K[X]f_i + K[X]f_j = 1$ for $i \neq j$.
(a) Show that

$$K[X]/(F(X)) \cong \sum_{i=1}^{n} K[X]/(f_i(X)^{e_i}).$$

(b) Show that $K[X]/(F(X))$ is Noetherian and has dimension 0.