Math 531 Problem Set #13 Due 12/08/00

1. Let $\mathcal{O}_L$ be ring of integers in a quadratic number field L. Let $p \mid \Delta(\mathcal{O}_L/\mathbb{Z})$. Show that

$$\left(\frac{\mathrm{N}_{L/\mathbb{Q}}(\alpha)}{p}\right) = 1 \text{ or } 0$$

for every $\alpha \in \mathcal{O}_L$.

2. Let $\mathcal{O}_L$ be ring of integers in a quadratic number field L. Suppose that there is some $p \mid \Delta(\mathcal{O}_L/\mathbb{Z})$ such that $p \equiv 3 \pmod 4$. Show that there does not exist any $\alpha \in \mathcal{O}_L$ such that $\mathrm{N}(\alpha) = -1$.

3. Let $\mathbb{P}^n_{\mathbb{Q}}$ be the set

$$\{(x_0, \ldots, x_n) \mid x_i \in \mathbb{Q}, (x_0, \ldots, x_n) \neq (0, \ldots, 0)\} / \sim$$

where $\sim$ is the equivalence relation

$$(x_0, \ldots, x_n) \sim (y_0, \ldots, y_n)$$

if there exists $\lambda \in \mathbb{Q}^*$ for which $x_i = \lambda y_i$ for $i = 0, \ldots, n$. We define the height function H as

$$H(x_0, \ldots, x_m) = \left(\prod_{p \text{ prime}} \max_i(|x_i|_p) \right) \max_i(|x_i|)$$

where $|x_i|_p = p^{-v_p(x)}$ for v_p is the p-adic valuation, when $x_i \neq 0$ and $|x_i|_p = 0$ if $x_i = 0$. Show that H is well-defined with respect to $\sim$ defined above.

4. Show that $|x + y| = \max(|x|, |y|)$ for a nonarchimedean $|\cdot|$ when $|x| \neq |y|$.

5. Write out the 2-adic representation for $1/3$, using 0 and 1 as representatives for the residue classes of $\mathbb{Z}_2/2\mathbb{Z}_2$.

6. Ex. 1, p. 99.

7. Ex. 2, p. 99.

8. Ex. 4, p. 99.