Math 531 Problem Set #11 Due 11/17/04

1. Show that $|\,\mathrm{Cl}(\mathbb{Z}[\frac{1+\sqrt{-7}}{2}])| = 1$.
2. Show that $|\,\mathrm{Cl}(\mathbb{Z}[\frac{1+\sqrt{29}}{2}])| = 1$.
3. Show that $|\,\mathrm{Cl}(\mathbb{Z}[\frac{1+\sqrt{-19}}{2}])| = 1$.
4. Use your result from #3 to find all the integers solutions x and y to the equation $x^2 + 19 = y^3$, the equation we studied the first day of class.
5. Let $d < -1$ be a squarefree number congruent to 3 (mod 4). Show that the prime ideal $\mathcal{P} \subseteq \mathbb{Z}[\sqrt{d}]$ with $\mathcal{P} \cap \mathbb{Z} = 2$ has order 2 in the class group $\mathrm{Cl}(\mathbb{Z}[\sqrt{d}])$. [Hint: You may want to try $d = -5$ separately from the other cases.]
6. Calculate the class group of $\mathbb{Z}[\sqrt{-5}]$.
7. Suppose that $d < -7$ is a squarefree number that is congruent to $1 \pmod 8$. Show that $|\,\mathrm{Cl}[\mathbb{Z}[\frac{1+\sqrt{d}}{2}]| \neq 1$. [Hint: Look at one of the primes lying over 2].
8. Let $d < -1$ be composite and squarefree. Let

$$\omega = \begin{cases} \sqrt{d} & : \quad d \equiv 2, 3 \pmod 4 \\ \frac{1+\sqrt{d}}{2} & : \quad d \equiv 1 \pmod 4 \end{cases}$$

Show that $|\,\mathrm{Cl}(\mathbb{Z}[\omega])| \neq 1$. [Hint: Look at the prime in $\mathbb{Z}[\omega]$ lying over the smallest positive prime factor of d. You may want to treat the case $d = -6$ separately from the others.]