Math 513 Tom Tucker NOTES FROM CLASS 9/29

We were proving the following:

Theorem 12.1. Let $L \supseteq K$ be a finite extension of fields. Then the bilinear form $(x, y) = T_{L/K}(xy)$ is nondegenerate $\Leftrightarrow L$ is separable over K.

Proof. (\Rightarrow) We did last time.

 (\Leftarrow) We will denote $T_{L/K}(xy)$ as (x, y). Recall the following: Choosing a basis m_1, \ldots, m_n and writing x and y as vectors in terms of the m_i we can write

 $\mathbf{x}A\mathbf{y}^T$

for some matrix A. The matrix A is given by $[a_{ij}]$ where $a_{ij} = (m_i, m_j)$ since we want

$$\left(\sum_{i=1}^{n} r_{i}a_{i}, \sum_{j=1}^{n} s_{j}a_{j}\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} r_{i}s_{j}(a_{i}, a_{j}).$$

It is easy to see that that the form will be nondegenerate if and only if A is invertible, since $A\mathbf{y} = 0$ if and only (x, y) = 0 for every $y \in L$.

Now, since L is separable over K, we can write $L = K(\theta)$ for $\theta \in L$ and use $1, \theta, \ldots, \theta^{n-1}$ as a basis for L over K. Then we can write the matrix $A = [a_{ij}]$ above with

$$a_{ij} = (\theta^i, \theta^j) = \mathcal{T}_{L/K}(\theta^{i+j}).$$

It isn't too hard to calculate these coefficients explicitly. In fact, if $\theta_1, \ldots, \theta_n$ are the roots of the minimal polynomial of θ , then

$$T_{L/K}(\theta) = \sum_{\ell=1}^{n} \theta_{\ell},$$

from what we proved earlier. Similarly, we have

$$\mathcal{T}_{L/K}(\theta^{i+j}) = \sum_{\ell=1}^{n} \theta_{\ell}^{i+j}.$$

There is a trick to finding the determinant of such a matrix. Recall the van der Monde matrix in $V := V(\theta_1, \ldots, \theta_n)$. It is the matrix

$$\begin{pmatrix} 1 & \cdots & 1\\ \theta_1 & \cdots & \theta_n\\ \cdots & \cdots & \cdots\\ \theta_1^n & \cdots & \theta_n^n \end{pmatrix}$$

The determinant of this matrix is

$$\det(V) = \prod_{i < j} (\theta_i - \theta_j)$$

It is easy to check that $VV^T = A$ (a messy but easy calculation). Thus,

$$\det(A) = \det(V) \det(V^T) = \det(V)^2 = \left(\prod_{i < j} (\theta_i - \theta_j)\right)^2 \neq 0,$$

since $\theta_i \neq \theta_j$ for $i \neq j$ and we are done.

Now, given a bilinear from (x, y) on a vector space W, we get a map from $\psi : W \longrightarrow W^*$, where W^* is the dual of W by sending $x \in W$ to the map f(y) = (x, y). When the form is nondegenerate this map is injective. Thus, by dimension counting, when W is finite dimensional and the form is nondegenerate, we get an isomorphism of vector spaces. In particular, we can do the following. Let u_1, \ldots, u_n be a basis for Wover V. Then for each u_i , there is a map $f_i \in W^*$ such that $f_i(u_j) = \delta_{ij}$ where δ_{ij} is the Kronecker delta, which means that $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if i = j. Since $f_i(x) = (v_j, x)$ for some $v_j \in W$, we obtain a dual basis v_1, \ldots, v_n with the property that

$$(v_i, u_j) = \delta_{ij}.$$

Thus, we have the following.

Theorem 12.2. (Dual basis theorem) Let $L \supseteq K$ be a finite, separable extension of fields. Let u_1, \ldots, u_n be basis for L as a K-vector space. Then there is a basis v_1, \ldots, v_n for L as a K-vector space such that

$$T_{L/K}(v_i, u_j) = \delta_{ij}.$$

Proof. Since $(x, y) = T_{L/K}(xy)$ is a nondegenerate bilinear form on L (considered as a K-vector space), we may apply the discussion above.

Definition 12.3. Let $L \supseteq K$ be a separable field extension. Let M be a submodule of L. We define M^{\dagger} to be set

$$\{x \in L \mid T_{L/K}(xy) \in A \text{ for every } y \in M\}$$

Remark 12.4. It is clear that $M \subseteq N \Rightarrow M^{\dagger} \supseteq N^{\dagger}$, by definition of the dual module.

Lemma 12.5. Let M be an A-submodule of L for which

$$M = Bu_1 + \dots + Bu_n$$

for u_1, \ldots, u_n a basis for L over K. Then M^{\dagger} is equal to $Bv_1 + \cdots + Bv_n$ for v_1, \ldots, v_n a dual basis for u_1, \ldots, u_n with respect to the bilinear form induced by the trace.

Proof. Let $x \in L$. Then $x \in M^{\dagger}$ if and only if $T_{L/K}(xu_i) \in A$ for each u_i . Writing x as $\sum_{i=1}^n \alpha_i v_i$ with $\alpha_i \in K$, we see that $T_{L/K}(xu_i) = \alpha_i$, so $T_{L/K}(xu_i) \in R$ if and only if $\alpha_i \in R$. This completes our proof. \Box

Theorem 12.6. Let A be a Dedekind domain with field of fractions K and let $L \supseteq K$ be a finite, separable extension of fields. Let B be the integral closure of A in L. Then B is Dedekind.

Proof. We already know that B is 1-dimensional, integrally closed, and an integral domain. We need only show that it is Noetherian.

Then $B \subset B^{\dagger}$ since B is integral over A (recall B integral over A means that the coefficients of the minimal polynomial for B over A are all in A). Now, we choose a basis u_1, \ldots, u_n for L over K. I claim that we can choose the u_i to be in B. This is because for any $u \in L$ we have

$$u^{m} + \frac{x_{m-1}}{y_{m-1}}u^{m-1} + \dots + \frac{x_{0}}{y_{0}} = 0$$

with x_i and y_i in A. Replacing u with $u' = \prod_{i=1}^m y_i$ and multiplying m

through by $(\prod_{i=1}^{m} y_i)^m$ converts this into an integral monic equation in u' as we've seen before. Thus, we can take our basis u_i , replace each u_i with a multiple of u_i and still have a basis. Let v_1, \ldots, v_n be a dual basis for u_1, \ldots, u_n with respect to the trace form. Then the A-module generated by the v_i contains B^{\dagger} . So we have

$$B \subseteq B^{\dagger} \supseteq Av_1 + \dots + Av_n$$

which implies that B is contained in a finitely generated A-module, which in turn implies that B is Noetherian as an A-module. Hence, B is Noetherian as a B-module and is a Noetherian ring.