Math 531 Tom Tucker
NOTES FROM CLASS 9/22

Note about the homework

The following is an immediate consequence of Problems 5 and 6 from the homework you just handed in.

Theorem 9.1. *Let R be a Noetherian integral domain of dimension 1. Then the following are equivalent*

(1) *R is integrally closed;*
(2) *R is a Dedekind domain.*

Proof. Since $R = \cap_{\mathcal{P}\neq 0} R_{\mathcal{P}}$ by Problem 6, it follows from Problem 5 that R is integrally closed if and only if each $R_{\mathcal{P}}$ is integrally closed. $\square$

A note on definitions: Fractional ideals are not generally always assume to be finitely generated. So here's what we have from last time with this convention.

Lemma 9.2. *Let J be a finitely generated fractional ideal of an integral domain R with field of fractions K and let S be a multiplicative set S in R not containing 0. Then $S^{-1}R(R : J) = (S^{-1}R : S^{-1}RJ)$.*

All invertible ideals are automatically finitely generated, though.

Lemma 9.3. *Let J be a fractional ideal of an integral domain R. Then J is invertible $\Leftrightarrow$ J is finitely generated and $R_{\mathcal{M}}J$ is an invertible fractional ideal of $R_{\mathcal{M}}$ for every maximal ideal $\mathcal{M}$ of R.*

Proof. ($\Rightarrow$) Let J be an invertible ideal ideal of R. Then we can write

$$\sum_{i=1}^{k} n_i m_i = 1$$

with $n_i \in (R : J)$. Since $n_i J \in R$ for each i, we can write any $y \in J$ as $\sum_{i=1}^{k}(n_i y)m_i = y$, so the m_i generate J. Hence, J is finitely generated. Let $\mathcal{M}$ be a maximal ideal of R. Since we can write $J(R : J) = R$ we must have $R_{\mathcal{M}}(J(R : J)) = R_{\mathcal{M}}$, so $(R_{\mathcal{M}}J)(R_{\mathcal{M}}(R : J)) = R_{\mathcal{M}}$, so $R_{\mathcal{M}}J$ is invertible

($\Leftarrow$) For any ideal J, we can form $J(R : J) \subseteq R$ (not necessarily equal to R). This will be an ideal I of R. Let $\mathcal{M}$ be a maximal ideal of R. Since J is finitely generated by assumption, we can apply the Lemma immediately above to obtain $(R_{\mathcal{M}} : R_{\mathcal{M}}J) = R_{\mathcal{M}}(R : J)$. Hence, we have $R_{\mathcal{M}}J(R : J) = R_{\mathcal{M}}$. Thus the ideal $I = J(R : J)$ is not contained in any maximal ideal of R. Thus, $I = R$ and J is invertible. $\square$

Theorem 9.4. *Let R be a a local integral domain of dimension 1. Then R is a DVR $\Leftrightarrow$ the maximal ideal $\mathcal{M}$ of R is invertible.*

Proof. ($\Rightarrow$) If J is a fractional ideal, then $xJ \subset R$ for some $x \in R$. Hence $xJ = Ra$ for some $a \in R$ since a DVR is PID. Thus, $J = Rax^{-1}$. Clearly $(R : J) = Ra^{-1}x$ and $J(R : J) = 1$, so J is invertible.

($\Leftarrow$) Since every nonzero ideal $I \subset R$ is invertible, every ideal of R is finitely generated, so R is Noetherian. Now, it will suffice to show that every nonzero ideal in R is a power of the maximal ideal $\mathcal{M}$ of R. The set of ideals I of R that are not a power of $\mathcal{M}$ (note: we consider R to $\mathcal{M}^0$, so the unit ideal is considered to be a power of $\mathcal{M}$) has a maximal element if it is not empty. Taking such a maximal element I, we see that $(R : \mathcal{M})I$ must not be invertible since if it had an inverse J, then $\mathcal{M}J$ would be an inverse for I. On the other hand, $(R : \mathcal{M})I \neq I$ since if $(R : \mathcal{M})I = I$, then $\mathcal{M}I = I$ which means that $I = 0$ by Nakayama's Lemma. Since $(R : \mathcal{M})I \supseteq I$ (since $1 \in (R : \mathcal{M})$), this means that $(R : \mathcal{M})I$ is strictly larger than I, contradicting the maximality of I. □

Now, we have the global counterpart.

Theorem 9.5. *Let R be a integral domain of dimension 1. Then R is a Dedekind domain $\Leftrightarrow$ every fractional ideal of R is invertible.*

Proof. ($\Rightarrow$) Let J be a fractional ideal of R. Then, for every maximal ideal $\mathcal{M}$, it is clear that $R_{\mathcal{M}}J$ is a fractional ideal of $R_{\mathcal{M}}$. Since $R_{\mathcal{M}}$ is a DVR, $R_{\mathcal{M}}J$ must be therefore be invertible for every maximal ideal $\mathcal{M}$. Moreover, J must be finitely generated since there is an $x \in K$ for which xJ is an ideal of R and every ideal of R is finitely generated since R is Noetherian. Therefore, J must be invertible by a Lemma 9.3.

($\Leftarrow$) Since every ideal of R is invertible, every ideal of R is finitely generated, so R is Noetherian. Now, since any maximal $\mathcal{P}$ is invertible, we write $(R : \mathcal{P})\mathcal{P} = R$. Localizing at $\mathcal{P}$ and treating things as $R_{\mathcal{P}}$ modules, we then obtain

$$R_{\mathcal{P}}(R : \mathcal{P})R_{\mathcal{P}}\mathcal{P} = R_{\mathcal{P}},$$

so from the theorem above $R_{\mathcal{P}}$ is a DVR and we are done.

□

Let's show that not only can every ideal I of a Dedekind domain R be factored uniquely, but so can every fractional ideal J of a Dedekind domain. Since every nonzero prime is invertible in R, we can write $\mathcal{P}^{-1} = (R : \mathcal{P})$ for maximal $\mathcal{P}$ (by the way nonzero prime means the same thing as maximal in a 1-dimensional integral domain of course).

Proposition 9.6. *Let R be a Dedekind domain. Then every fractional ideal J of R has a unique factorization as*

$$J = \prod_{i=1}^{n} \mathcal{P}_i^{e_i}$$

with all the $e_i \neq 0$.

Proof. To see that J has some factorization as above we note xJ is an ideal I in R. So if we factor Rx and I and write $J = (x)^{-1}I$, we have a factorization. To see that the factorization is unique we write

$$I = (\prod_{i=1}^{n} \mathcal{P}_i^{e_i})(\prod_{j=1}^{m} \mathcal{Q}_j^{-f_j})$$

with all the e_i and f_j positive and no $\mathcal{Q}_j$ equal to any $\mathcal{P}_i$. Let $I = \prod_{j=1}^{m} \mathcal{Q}_j^{f_j}$ Then JI^2 is an ideal of R with $JI^2 = (\prod_{i=1}^{n} \mathcal{P}_i^{e_i})(\prod_{j=1}^{m} \mathcal{Q}_j^{f_j})$. Since I^2 has a unique factorization and so does JI^2, so must J have a unique factorization. $\square$