Now, a breif interlude on geometry and normality. Let $F(X,Y)=0$ be a curve in the plane k^2 over an algebraically closed field k. We say that $F(X,Y)$ is *singular* at the point (a,b) is

$$\frac{\partial F}{\partial X}(a,b)=\frac{\partial F}{\partial Y}(a,b)=0.$$

In other words if the tangent vector to $F=0$ is 0 at (a,b), so that there is no notion of a tangent vector here. If the point (a,b) is not singular, we say that it is nonsingular.

Note that the primes $\mathcal{Q}$ of R correspond to points (a,b) such that $F(a,b)=0$. If $\mathcal{Q}$ corresponds to the point (a,b) then $\mathcal{Q}$ is simply the image of $k[X,Y](X-a)+k[X,Y](Y-b)$ in R.

Lemma 6.1. *Let $\mathcal{Q}$ be a nonzero prime in the ring*

$$R=k[X,Y]/F(X,Y).$$

Then

$$\dim_k \mathcal{Q}/\mathcal{Q}^2=1$$

if and only if the point (a,b) corresponding to $\mathcal{Q}$ is nonsingular.

Proof. Let $\mathcal{P}$ be prime in $k[X,Y]$ generated by $(X-a)$ and $(Y-b)$. Let θ be the map from $k[X,Y]$ to k^2 given by

$$\theta(G)=(\frac{\partial G}{\partial X}(a,b),\frac{\partial G}{\partial Y}(a,b)).$$

Then $\theta(X-a)=(1,0)$ and $\theta(Y-b)=(0,1)$, so the rank of the image of $\mathcal{P}$ is 2. It is easy to see that $\mathcal{P}^2$ is in the kernel of this map. So θ induces an isomorphism between $\mathcal{P}/\mathcal{P}^2$ and k^2. Now we have

$$\mathcal{Q}/\mathcal{Q}^2\cong(\mathcal{P}/(\mathcal{P}^2+F(X,Y)),$$

as a k-vector space since

$$\mathcal{P}^2+F(X,Y)=\phi^{-1}(\mathcal{Q}^2)$$

where ϕ is the quotient map from $k[X,Y]$ to R. Counting dimensions we have

$$\dim_k \mathcal{P}/\mathcal{P}^2=2$$

if $\theta(F)=0$ and and

$$\dim_k \mathcal{P}/\mathcal{P}^2=1$$

otherwise. □

Lemma 6.2. *(Later in class) We have*

$$\mathcal{Q}/\mathcal{Q}^2\cong R_{\mathcal{Q}}\mathcal{Q}/(R_{\mathcal{Q}}\mathcal{Q})^2.$$

Lemma 6.3. *Let R be a ring that has direct sum decomposition*

$$R = \bigoplus_{j=1}^{n} R_j.$$

Then every ideal in $I \subset R$ can be written as

$$I = \bigoplus_{j=1}^{n} I_j$$

for ideals $I_j \subset R_j$. If $\mathcal{P}$ is a prime of R then there is some j for which we can write

$$\mathcal{P} = \bigoplus_{\ell \neq j} R_\ell \bigoplus \mathcal{P}_j$$

Proof. We can view $R = \bigoplus_{j=1}^{n} R_j$ as the set of

$$(r_1, \ldots, r_n)$$

with $r_j \in R_j$. Let p_j be the usual projection from R onto its j-th coordinate and let i_j be the usual embedding of R_j into R obtained by sending $r_j \in R_j$ to the element of R with all coordinates 0 except for the j-th coordinate which is set to r_j. Since an ideal I of R must be a $i_j(R_j)$ module, the set of $p_j(r)$ for which $r \in I$ must form an ideal R_j ideal, call it I_j. It is easy to see that $I_j = p_j(I)$. Certainly, $I \subset \bigoplus p_j(I)$. Since we can multiply anything in I by $(0, \ldots, 1_j, 0, \ldots, 0)$ we see that $i_j p_j(I) \subset I$. Hence $\bigoplus p_j(I) \subset I$, and we are done with our description of ideals of $\bigoplus_{j=1}^{n} R_j$. For prime ideals, we note that if $\mathcal{P}$ is a prime then $(a_1, \ldots, a_n)(b_1, \ldots, b_n) \in \mathcal{P}$ implies that $a_j b_j \in p_j(\mathcal{P})$ for each j, so $p_j(\mathcal{P})$ must be a prime of R_j or all of R_j. Suppose we had $k \neq j$ with $p_j(\mathcal{P}) \neq R_j$ and $p_k(\mathcal{P}) \neq R_k$. Then choosing $a_j \in p_j(\mathcal{P})$, $a_k \in p_k(\mathcal{P})$ and $b_j \notin p_j(\mathcal{P})$, $b_k \notin p_k(\mathcal{P})$, we see that

$$(i_j(a_j) + i_j(b_k))(i_j(b_j) + i_k(a_k)) \in \mathcal{P},$$

but $(i_j(a_j)+i_j(b_k)), (i_j(b_j)+i_k(a_k)) \notin \mathcal{P}$, a contradiction, so $p_j(\mathcal{P}) = R_j$ for all but one j. Thus

$$\mathcal{P} = \bigoplus_{\ell \neq j} R_\ell \bigoplus \mathcal{P}_j$$

for some prime $\mathcal{P}_j$ of R_j. □

Corollary 6.4. *Let R be a Noetherian ring in which every prime ideal is maximal. Then R has only finitely many prime ideals $\mathcal{P}_1, \ldots, \mathcal{P}_n$ and can be written as*

$$R \cong \bigoplus_{j=1}^{n} R/\mathcal{P}_i^{w_i}.$$

Proof. Since R is Noetherian, there are prime ideals $\mathcal{P}_i$ such that $\prod_{j=1}^{n} \mathcal{P}_i^{w_i} = 0$ (remember that we can make the product be contained in 0 and 0 is the only element in $R0$). Then the natural map

$$R \longrightarrow \bigoplus_{j=1}^{n} R/\mathcal{P}_i^{w_i}$$

is surjective with kernel 0, hence it is an isomoprhism. Within each factor $R/\mathcal{P}_i^{w_i}$, the only prime ideal is the image of $\mathcal{P}_i$ under the quotient map ϕ, since the image of any other prime under ϕ is all of $R/\mathcal{P}_i^{w_i}$ by the Lemma above. Hence, $\phi(\mathcal{P}_i)$ is the only prime in $R/\mathcal{P}_i^{w_i}$. By the Lemma above, the only primes in R are of the form $\bigoplus_{\ell \neq j} R \bigoplus \phi(\mathcal{P}_i)$. □

Corollary 6.5. *Let R be a Noetherian ring of dimension 1. Then every nonzero ideal I is contained in finitely many prime ideals $\mathcal{P}$.*

Proof. Every prime ideal in R/I is maximal, so the proposition above applies. □

Lemma 6.6. *Let R be a integral domain, let $\mathcal{M}$ be a maximal ideal of R, let $n \geq q$, and let ϕ the quotient map $\phi : R \longrightarrow R/\mathcal{M}^n$ be the quotient map. Then $\phi(s)$ is a unit in $R/\mathcal{M}^n$ for every $s \in R \setminus \mathcal{M}$.*

Proof. Since $\mathcal{M}$ is maximal, we can have $Rs + \mathcal{M} = 1$ for $s \notin \mathcal{M}$. Thus, we can write $ax + m = 1$ for $a \in R$ and $m \in \mathcal{M}^n$ using facts about coprime ideals proved earlier. Thus $ax = 1 \pmod{\mathcal{M}^n}$, so $\phi(ax) = 1$. □