Math 531 Tom Tucker

Let's finish off the Proposition from last time.

Proposition 7.1. *Let R be a Noetherian local domain of dimension 1 with maximal ideal $\mathcal{M}$ and with $R/\mathcal{M} = k$ its residue field. Then the following are equivalent*

(1) *R is a DVR;*
(2) *R is integrally closed;*
(3) *$\mathcal{M}$ is principal;*
(4) *there is some $\pi \in R$ such that every element $a \in R$ can be written uniquely as $u\pi^n$ for some unit u and some integer $n \geq 0$.*
(5) *every nonzero ideal is a power of $\mathcal{M}$;*

Proof. ($2 \Rightarrow 3$) Let $a \in \mathcal{M}$. There is some n for which $\mathcal{M}^n \subset (a)$ (by"Poor Man's Factorization" in Noetherian rings) but $\mathcal{M}^{n-1}$ is not contained in (a) (note $n-1$ could be zero). Let $b \in \mathcal{M}^{n-1} \setminus (a)$ and let $x = a/b$. We can show that $\mathcal{M} = Rx$. This is equivalent to showing that $x^{-1}\mathcal{M} = R$. Note that since (b) is not in (a), $b/a = x^{-1}$ cannot be in R. Hence, it cannot be integral over R. By Cayley-Hamilton, $x^{-1}\mathcal{M} \neq \mathcal{M}$ since $\mathcal{M}$ is finitely generated as an R-module and $x^{-1} \notin R$ and R is integrally closed. Since $x^{-1}\mathcal{M}$ is an R-module and $x^{-1}\mathcal{M} \subset A$ (this follows from the fact that $b\mathcal{M} \subset \mathcal{M}^n \subset (a)$), this means that $x^{-1}\mathcal{M}$ is an ideal of R not contained in $\mathcal{M}$. So $x^{-1}\mathcal{M} = R$, as desired. □

One more criterion related to being a DVR.

Proposition 7.2. *LEt A be a Noetherian local ring with maximal ideal $\mathcal{M}$. Supppose that*

$$Rx_1 + \cdots + Rx_n + \mathcal{M}^2 = \mathcal{M},$$

for $x_i \in R$. Then $Rx_1 + \cdots + Rx_n = \mathcal{M}$.

Proof. Let $N = \mathcal{M}/(Rx_1 + \ldots Rx_n$. Then $\mathcal{M}N = N$, so $N = 0$ by Nakayama's lemma, since N is finitely generated. □

Corollary 7.3. *Let A be a Noetherian local ring. Let $\mathcal{M}$ be its maximal ideal and let k be the residue field $A/\mathcal{M}$. Then*

$$\dim_k \mathcal{M}/\mathcal{M}^2 = 1$$

if and only if $\mathcal{M}$ is principal

Proof. One direction is easy: If $\mathcal{M}$ is generated by π, then $\mathcal{M}/\mathcal{M}^2$ is generated by the image of π modulo $\mathcal{M}^2$. To prove the other direction, suppose that $\mathcal{M}/\mathcal{M}^2$ has dimension 1. Then we can write $\mathcal{M} = Ra + \mathcal{M}^2$ for some $a \in \mathcal{M}$. Then the module $M = \mathcal{M}/a$ has the property that $\mathcal{M}M = M$, since any element in M can be written as $ca + d$ for $c \in R$ and $d \in \mathcal{M}^2$. By Nakayama's lemma, we thus have $M = 0$, so $\mathcal{M} = Ra$. □

Proposition 7.4. *Let R be a domain and let $S \subset R$ be a multiplicative subset not containing 0. Let $b \in K$, where K is the field of fractions of R. Then b is integral over $S^{-1}R \Leftrightarrow sb$ is integral over R for some $s \in S$.*

Proof. If b is integral over $S^{-1}R$, then we can write

$$b^n + \frac{a_{n-1}}{s_{n-1}} b^{n-1} + \cdots + \frac{b_0}{s_0} = 0.$$

Letting $s = \prod_{i=0}^{n-1} s_i$ and multiplying through by s^n we obtain

$$(sb)^n + a'_{n-1}(sb)^{n-1} + \cdots + a'_0 = 0$$

where

$$a'_i = s^{n-i-1} \prod_{\substack{j=1 \\ j \neq i}}^{n} s_i a_i$$

which is clearly in R. Hence sb is integral over R. Similarly, if an element sb with $b \in S^{-1}R$ and $s \in S$ satisfies an equation

$$(sb)^n + a_{n-1}(sb)^{n-1} + \cdots + a_0 = 0,$$

with $a_i \in R$, then dividing through by s^n gives an equation

$$b^n + \frac{a_{n-1}}{s} b^{n-1} + \cdots + \frac{a_0}{s^n},$$

with coefficients in $S^{-1}R$.

□

Corollary 7.5. *If R is integrally closed, then $S^{-1}R$ is integrally closed.*

Proof. When R is integrally closed, any b that is integral over R is in R. Since any element $c \in K$ that is integral over $S^{-1}R$ has the property that sc is integral over R for some $s \in S$, this means that $sc \in R$ for some $s \in S$ and hence that $c \in S^{-1}R$.

□

Lemma 7.6. *Let $A \subseteq B$ be domains and suppose that every element of B is algebraic over A. Then for every ideal nonzero I of B, we have $I \cap A \neq 0$.*

Proof. Let $b \in A$ be nonzero. Since b is algebraic over A and $b \neq 0$, we can write

$$a_n b^n + \cdots + a_0 = 0,$$

for $a_i \in A$ and $a_0 \neq 0$. Then $a_0 \in I \cap \mathbb{Z}$. □

Theorem 7.7. *Let α be an algebraic number that is integral over $\mathbb{Z}$. Suppose that $\mathbb{Z}[\alpha]$ is integrally closed. Then $\mathbb{Z}[\alpha]$ is a Dedekind domain.*

Proof. Since $\mathbb{Z}[\alpha]$ is a finitely generated $\mathbb{Z}$-module, any ideal of $\mathbb{Z}[\alpha[$ is also a finitely generated $\mathbb{Z}$-module. Hence, any ideal of $\mathbb{Z}[\alpha]$ is finitely generated over $\mathbb{Z}[\alpha]$, so $\mathbb{Z}[\alpha]$ is Noetherian. Let $\mathcal{Q}$ be a prime in $Z[\alpha]$. Then, $\mathcal{Q} \cap \mathbb{Z}$ is a prime ideal (p) in $\mathbb{Z}$. Hence, $\mathbb{Z}[\alpha]/\mathcal{Q}$ is a quotient of $\mathbf{F}_p[X]/f(X)$ where $f(X)$ is the minimal monic satisfied by α. Since $\mathbf{F}_p[X]/f(X)$ has dimension 0 (Exercise 7 on the homework), this implies that $\mathbb{Z}[\alpha]/\mathcal{Q}$ is a field so $\mathcal{Q}$ must be maximal. $\square$

Remark 7.8. The rings we deal with will *not* in general have this form.