Before, going on, let's prove one more thing about integrality... do Prop. 2.5 here.

Proposition 3.1. *(Prop. 2.5 from Janusz) Let R be a domain with field of fractions K and let L be an algebraic extension of K. Let $b \in L$ and let $f(X)$ be the minimal polynomial for b that has coefficients in K and leading coefficient 1. Then, the coefficients of f are integral over R whenever b is integral over R. In particular, if R is integrally closed in K and b is integral over R, then the coefficients of f are in R.*

Proof. Suupose that b is integral over R. We can write

$$f(X) = (X - b_1)(X - b_2) \cdots (X - b_n),$$

by extending L to some field E over which f splits. Note that any polynomial satisfied by b is divisible by f in $K[X]$, so if b satisfies an integral polynomial with coefficients in R, so do all of the other b_i. Hence, if b is integral then so are all of the b_i. The coefficients of f are all in the ring $R[b_1, \ldots, b_n]$, so this also means that the coefficients of f are integral over R as desired. Now, since these coefficients are also in K, they are actually in R if R is integrally closed. $\square$

So, to check if something is integral, all we have to do is check its minimal polynomial. Example, let $\alpha = \sqrt{11}/7$. Its minimal polynomial is $X^2 - 11/49$ which isn't integral over $\mathbb{Z}$, so we're done.

Last time we were in the process of defining Noetherian rings. Recall...

Definition 3.2. A ring A is said to be *Noetherian* if it satisfies the *ascending chain condition* which states that if there is a sequence of ideals I_m such that

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_m \subseteq \ldots$$

then there is an N such that for all $n \geq N$, we have $I_n = I_{n+1}$. In other word, the chain becomes stationary.

A quick word on maximality: an ideal I is maximal if there is no larger proper ideal J containing I. Maximal ideals are usually written as $\mathcal{M}$.

Lemma 3.3. *Let A be a Noetherian ring. Any subset $\mathcal{S}$ of ideals of A has a maximal element (here maximal means that there is no strictly larger ideal $I' \supset I$ in $\mathcal{S}$).*

Proof. Let $I_0 \in \mathcal{S}$. If I is not maximal in $\mathcal{S}$ there is a larger ideal $I_1 \in \mathcal{S}$ containing I_0. Similarly, if I_1 is not maximal there is a larger ideal $I_2 \in \mathcal{S}$ containing it and so on, so we have an ascending chain of ideals

$$I_0 \subseteq I_1 \subseteq \cdots \subseteq I_m \subseteq \dots,$$

which means that there is some N such that for all $n \geq N$, we have $I_n = I_{n+1}$ Then I_N is a maximal element of $\mathcal{S}$. □

Proposition 3.4. *R is Noetherian $\Leftrightarrow$ every ideal of R is finitely generated.*

Proof. ($\Rightarrow$) Let J be an ideal and let $\mathcal{S}_J$ be set of all finitely generated ideals contained in J. This set is nonempty since for any $a \in J$, the ideal $Ra \subset J$ is finitely generated. Let I be a maximal element of $\mathcal{S}_J$. If I is not equal to J, then there is some $b \in J$ such that $b \notin I$. But $I + Ra$ is also finitely generated and strictly larger than I, so this is impossible. Thus, $I = J$ and j is finitely generated.
($\Leftarrow$) Let

$$I_0 \subseteq I_1 \subseteq \cdots \subseteq I_m \subseteq \dots,$$

be an ascending chain of ideals. Then $\cup_{j=0}^{\infty} I_j$ is an ideal (easy to check) and is finitely generated, by, say, the set $a_1, \dots, a_\ell$. Each a_i is in some I_j so there is an I_N containing all of the a_i. Thus, $I_N = \cup_{j=0}^{\infty} I_j$ and $I_{n+1} = I_n$ for every $n \geq N$.

□

Recall an ideal $\mathcal{P}$ is said to be prime if $ab \in \mathcal{P}$ implies that either $a \in \mathcal{P}$ or $b \in \mathcal{P}$.

Definition 3.5. The *dimension* of a ring is the largest n for which there exists a chain of prime ideals

$$\mathcal{P}_0 \subset \mathcal{P}_1 \subset \cdots \subset \mathcal{P}_n,$$

where the $\mathcal{P}_i$ are prime ideals and $\mathcal{P}_i \neq \mathcal{P}_{i+1}$ (for $i = 1, \dots, n-1$).

Not all rings are finite dimensional, e.g. $k[(x_i)_{i=1}^{\infty}]$. This ring isn't Noetherian either. But furthermore, not all Noetherian rings are finite dimensional. However, all local Noetherian rings *are* finite dimensional.

Now, let's define localization... Let A be a domain and let $S \subset A$ be closed under multiplication and suppose that $0 \notin S$. Then, we can form a the ring $S^{-1}A$ which is the set of fraction of the form

$$\frac{a}{s}$$

where $a \in A$ and $s \in S$ subject to the equivalence relation

$$\frac{a}{s} = \frac{b}{t}$$

if $at = bs$. It is easy to check that is well-defined, e.g. that if $at = bs$ and $a't' = b's'$ then

$$\frac{a}{s}\frac{b}{t} = \frac{a'}{s'}\frac{b'}{t'}$$

and

$$\frac{a}{s} + \frac{b}{t} = \frac{a'}{s'} + \frac{b'}{t'}.$$

Note furthermore that s/s serves as 1 and that $0/s$ serves as 0. Also there is a natural map sending A into $S^{-1}A$ by fixing $s \in S$ and sending a to as/s.

Remark 3.6. Note that we need to change things slightly when S contains zero divisors. We say that $a/s = a'/s'$ if there exists some $t \in S$ such that $tas' = ta's$.

On the other hand, when A is a domain the map $A \longrightarrow S^{-1}A$ is always injective. Since $a/1 = 0/t$ implies that $at = 0$ which implies that $a = 0$.

When $\mathcal{P}$ is a prime elements than $A \setminus \mathcal{P}$ is multiplicatively closed set since $a, b \notin \mathcal{P}$ implies that $ab \notin \mathcal{P}$. This is the most important example of localization and in this case $S^{-1}A$ is written as $A_{\mathcal{P}}$. Examples...

Example 3.7. Localizing $\mathbb{Z}$ at the ideal (p) for p a prime number we get the set of elements of $\mathbb{Q}$ that can be written as a/s where s is not divisible by p.

Some more notation...people frequently write R_S rather $S^{-1}R$ simply because it is easier to write (for example, Janusz does this).

Some theorems from the book about localization. A quick note on prime ideals: we do not consider the whole ring R to be a prime ideal.

Lemma 3.8. *Let R be an integral domain. Let S be a multiplicative subset of R that does not contain 0. There is a bijection between the primes in R that do not intersect S and the primes in R_S.*

Proof. Denote the map from R into R_S as ϕ. Every prime ideal $\mathcal{Q}$ in R_S pulls back to a prime ideal $\phi^{-1}(\mathcal{Q})$ of R. We also see that an ideal $\mathcal{P}$ in R is equal to $\phi^{-1}(\mathcal{Q})$ for some $\mathcal{Q}$ in R_S if $\phi(\mathcal{P})$ is a prime ideal and $\phi^{-1}(\phi(\mathcal{P})) = \mathcal{P}$. Now, if there is some $s \in S \cap \mathcal{P}$, then clearly $R_S\mathcal{P} = 1$, since $\frac{1}{s}s = 1$. So it only remains to show that if $\mathcal{P}$ is a prime that doesn't intersect S, then $R_s\mathcal{P}$ is a prime ideal. It is easy to see that $R_s\mathcal{P}$ consists of all a/s for which $a \in \mathcal{P}$ and $s \in S$. Now, suppose that

$$\frac{x}{t}\frac{y}{t'} = \frac{a}{s}$$

for $x, y \in R$, $t, t' \in S$ and $a/s \in R_S$. Then $xys = att'$, so $xy \in \mathcal{P}$ (since $s \notin \mathcal{P}$, so either x or y is in $\mathcal{P}$, so either x/t or y/t' is in $R_S\mathcal{P}$. Thus, $R_S\mathcal{P}$ is indeed a prime ideal. $\square$

Forming $S^{-1}R$ is called *localizing* R. We define a local ring to be a ring with only one maximal ideal, e.g. $\mathbb{Z}_{(p)}$ is a local ring.