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NOTES FROM CLASS 12/10

We will show that if L is an algebraic extension of a field K_v that is complete with respect to an absolute value coming from a discrete valuation v, that there is in fact only *one* way of extending $|\cdot|_v$ to an absolute value on L. First, we'll need a definition of a metric on a vector space.

Definition 43.1. Let K be a field with an absolute value $|\cdot|_v$ and let W be a vector space over K A *v-metric* $\|\cdot\|$ on W is function $\|\cdot\| : W \longrightarrow \mathbb{R}$ such that

(1) $\|x\| \geq 0$ for every $x \in L$ and $|x| = 0$ if and only if $x = 0$.
(2) $\|ax\| = |a|\|x\|$ for every $a \in K$ and $x \in W$.
(3) (Triangle inequality) $\|x + y\| \leq \|x\| + \|y\|$ for any $x, y \in W$

Definition 43.2. We'll say that two metrics $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if there exist constants C_1 and C_2 such that

$$C_1\|x\|_1 \leq \|x\|_2 \leq C_2\|x\|_1$$

for every $x \in W$.

When W has finite dimension n with basis $e_1, \ldots, e_n$, we define

$$\|x_1e_1 + \ldots x_ne_n\|_{sup} = \max_i(|x_i|).$$

Then $\|\cdot\|_{sup}$ is clearly a metric. Moreover, for any Cauchy sequence in W, the coordinate x_i converge with respect to $|\cdot|_v$. Thus, it is clear that when K is complete with respect to v, W is complete with respect to $\|\cdot\|_{sup}$.

Lemma 43.3. *Let K_v be complete with respect to an absolute value $|\cdot|_v$ and let W be a finite dimensional vector space. Then any v metric $\|\cdot\|$ on W is equivalent to the metric $\|\cdot\|_{sup}$.*

Proof. It is easy to see that by the triangle inequality, we have

$$\|x_1e_1 + \ldots x_ne_n\| \leq \|x_1e_1\| + \ldots \|x_ne_n\| \leq n \max_i(\|e_i\|) \max_i(|x_i|).$$

Now, we need to show there is a C_1 such that $\|y\| \geq C_1\|y\|_{sup}$. Note that this is clearly true when $n = 1$, since in this case $\|y\| = \|y\|_{sup}$ for any y. Thus, we can proceed by induction on n. Suppose to the contrary this were not true - then for any i there would be some y_i such that

$$\|y_i\| \leq \frac{1}{i}\|y_i\|_{sup}.$$

Dividing through by $\|y_i\|_{sup}$, we can assume that $\|y_i\| = 1$. After reordering the e_j and throwing some of the y_i , we can assume that

the coefficient of e_1 in the expansion of y_i with respect to the basis $e_1, \ldots, e_n$ is 1. Then $c_i = y_i - e_1$ is in an $n-1$ dimensional vector space on which $\|\cdot\|$ is equivalent to $\|\cdot\|_{sup}$. For any i, j, we see that

$$\|c_i - c_j\| = \|y_i - y_j\| \leq \frac{1}{i} + \frac{1}{j},$$

so the c_j form a Cauchy sequence with respect to $\|\cdot\|$. Since $\|\cdot\|$ is equivalent to $\|\cdot\|_{sup}$ on the c_i, this means that the limit $\lim_i c_i$ exists and is in the space generated by $e_2, \ldots e_n$. Letting $c^* = \lim_i c_i$, we see that

$$\|c^* + e_1\| = \lim_i \|c_i + e_1\| \lim_i \|y_i\| = 0$$

but $c^* + e_1 \neq 0$, a contradiction of part (i) of the definition of a metric, so we have a contradiction. □

Theorem 43.4. *Let K_v be complete with respect to an absolute value $|\cdot|_v$ coming from the discrete valuation v. Let L be a finite separable extension of K_v. Then $|\cdot|_v$ extends uniquely to a absolute value $|\cdot|_w$ on L. Moreover, L is complete with respect to $|\cdot|_w$.*

Proof. We know that the prime lying over the maximal ideal in B_v are in one-to-one correspondence with absolute values $\|\cdot\|_w$ extending $\|\cdot\|_v$. Any absolute value on L as a field is also a metric on L as a K-vector space. Thus, any two absolute values on L extending v are equivalent as metrics. On the other hand, if $\mathcal{Q}_i \neq \mathcal{Q}_j$ are primes in C_v lying over B_v, then there exists an element $\pi \in \mathcal{Q}_i$ that isn't in $\mathcal{Q}_j$. If $\|\cdot|_w$ and $\|\cdot|_{w'}$ are the absolute values coming from $\mathcal{Q}_i$ and $\mathcal{Q}_j$ respectively, we see that for any $\epsilon > 0$, there is a suitably large power π^n of π for which $\|\pi^n\|_w < \epsilon$; we also know that $\|\pi^n\_w' = 1$, so $\|\cdot|_w$ and $\|\cdot\|_{w'}$ are not equivalent.

Since $\|\cdot\|_w$ is equivalent to the sup norm defined above and L is complete with respect to the sup norm, L must be complete with respect to $\|\cdot\|_w$, so we are done. □

Let K be a field with a discrete valuation on v; let B be the corresponding discrete valuation ring in K and let $\mathcal{P}$ be the maximal ideal in B. Let L be a finite separable extension of K; let C be the algebraic closure of B in L. We write

$$C\mathcal{P} = \mathcal{Q}_1^{e_1} \cdots \mathcal{Q}_m^{e_m}.$$

Let $f_i = [\mathcal{Q}_i : \mathcal{P}_1]$. Let $|\cdot|_w$ be the absolute value on L extending v that corresponds to $\mathcal{Q}_i$.

Proposition 43.5. *We have $[L_w : K_v] = e_i f_i$. This is called the local degree at w.*

Proof. Let B_v, C_w, $\mathcal{P}_v$, and $\mathcal{Q}_v$ denote the completions of B, C, $\mathcal{P}$, and $\mathcal{Q}_i$, respectively. Since C_w is a DVR, the only maximal ideal in C_w is $\mathcal{Q}_w$. Thus, C_w contains $C_{\mathcal{Q}_i}$, Since

$$C_{\mathcal{Q}_i}\mathcal{P} = \mathcal{Q}_i^{e_i},$$

we have

$$C_w\mathcal{P}_v = \mathcal{Q}_w^{e_i}.$$

We also have $C_w/\mathcal{Q}_w \cong C/\mathbb{Q}_i$ and $B_v/\mathcal{P}_v \cong B/\mathcal{P}$, so

$$f_i = [C_w/\mathcal{Q}_w : B_v/\mathcal{P}_v].$$

Thus,

$$[L_w : \mathbb{Q}_v] = e_i f_i.$$

$\square$