Math 531 Tom Tucker
NOTES FROM CLASS 12/08

**Lemma 42.1.** *Let $|\cdot|$ be an absolute value on a field $K$ such that $|1| = 1$. Then $|\cdot|$ is nonarchimedean $\Leftrightarrow$ $|n \cdot 1| \leq 1$ for every integer $n$.*

*Proof.* The $\Rightarrow$ direction follows immediately from the ultrametric inequality plus the fact that $n \cdot 1 = 1 + \ldots 1$. To prove the $\Leftarrow$ direction, let $x, y \in K$. We will show that $|x+y| \leq \max(|x|, |y|)$ assuming that $|n \cdot 1| \leq 1$ for every integer $n$. We may assume WLOG that $|x| > |y|$. Thus expanding $(x+y)^n$ out for an integer $n$, we have

$$\begin{aligned} |x+y|^n &\leq |\sum_{i=0}^{n} \binom{n}{i} ||x|^i |y|^{n-i} \\ &\leq \sum_{i=0}^{n} |x|^n \\ &\leq N|x|^n \end{aligned}$$

Taking $n$-th roots gives $|x+y| \leq \sqrt[n]{n}|x|$ for any $n$, so we must have

$$|x+y| \leq (\lim_{n\to\infty} \sqrt[n]{n})|x| = |x|.$$

□

**Proposition 42.2.** *Let $v$ be a discrete valuation on a field $K$ and let $L$ be a finite separable field extension of $K$. Let $B$ the set of $x$ in $K$ with $v(x) \geq 0$ and let $C$ be the integral closure of $B$ in $L$. Then the absolute values $|\cdot|_w$ on $L$ extending $|\cdot|_v$ are in one-to-one correspondence with the primes $\mathcal{P}$ in $\mathcal{O}_L$ lying over the maximal ideal $\mathcal{M}$ of $B$.*

*Proof.* Let $\mathcal{Q}_i$ be a prime lying above $\mathcal{M}$ in $C$. Since $C$ is Dedekind, the localization of $C$ is a DVR and there is a discrete valuation $w : L \longrightarrow \mathbb{Z} \cup \infty$ such that $C_{\mathcal{Q}_i}$ is the set of all $x \in L$ for which $v(x) \geq 0$. The valuation $|x|_w = e^{-w(x)}$ does not, however, restrict to $v$. To see this let $\gamma$ be a generator for $\mathcal{Q}_i$ in $C$ (since $C$ has finitely many primes, it is a PID), and let $\pi$ be a generator for $\mathcal{P}$ in $B$. Since

$$\mathcal{P}C = \mathcal{Q}_1^{e_1} \cdots \mathcal{Q}_i^{e_i} \cdots \mathcal{Q}_m^{e_m}$$

we can write $\pi = \gamma^{e_i} u$ for $u$ unit in $C_{\mathcal{Q}_i}$. Thus $w(\pi) = e_i$. To compensate for this, we let

$$|x|_w = e^{-\frac{1}{e_i} w(x)}$$

for $x \in L$ and obtain an absolute value on $L$ extending $|\cdot|_v$. It is clear that different $\mathcal{Q}_i$ give rise to different $|\cdot|_w$.

To see that these are the only absolute values on $L$ extending $|\cdot|_v$, we first observe that from the lemma above, we know that any absolute value $|\cdot|$ extending $|\cdot|_v$ must be nonarchimedean. Thus, the set of all $C_v$ of all $x \in L$ such that $w(x) \geq 0$ forms a ring. If $x \in L$ is integral, then writing

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$$

with $|a_i|_w = |a_i|_w \leq 1$, we see that $w(x) \geq 0$, since otherwise

$$|x^n + a_{n-1}x^{n-1} + \cdots + a_0|_w = w(x)^n > 1 > 0.$$

Now, the prime $\mathcal{P}$ of all $x \in C$ such that $|x|_w < 1$ must form a prime ideal of $C$ and cannot be the 0-ideal since it must contain $\mathcal{M}$. Thus, it must be one of the $\mathcal{Q}_i$. It follows that for all $x \in C_{\mathcal{Q}_i}$ we have $|x|_w \leq 1$ and that for any unit $u \in C_{\mathcal{Q}_i}$, we have $|u|_w = 1$. Let $\pi$ be a generator for $\mathcal{Q}_i$. Since any element of $x \in L$ can be written as $x = u\pi^n$, $|\cdot|_w$ is determined by its value on $\pi$, which was determined above. Thus $|\cdot|_w$ agrees with the absolute value coming from $\mathcal{Q}_i$ constructed above. $\square$