Math 531 Tom Tucker
NOTES FROM CLASS 12/03

There is one important difference between the p-adic absolute values and the ones coming from embedding L in to $\mathbb{C}$, the so-called *real* absolute values. This difference lies in a stronger form of the triangle inequality satisfied by the p-adic absolute values. Recall that a valuation $v : K^* \longrightarrow \mathbb{R}$ is a multiplicative map for which $v(x+y) \geq \min(v(x), v(y))$ for any $x, y \in K^*$. This last condition means that

$$|x+y| = e^{-(v(x)+v(y))} \leq e^{-\min(v(x),v(y))} \leq \max(|x|, |y|).$$

On the other hand, for the real valuation $|\cdot|$, we have, for example

$$|1+1| = 2 > \max(1,1).$$

A valuation v is called a discrete valuation if

$$v : K^* \longrightarrow \mathbb{Z} \subseteq \mathbb{R}$$

surjectively. By convention, we set $v(0) = \infty$.

Definition 40.1. If $|x+y| \leq \max(|x|, |y|)$ for every $x, y \in K$, then $|\cdot|$ is called an nonarchimedean valuation. Otherwise, it is called an archimedean valuation.

Example 40.2. Let $L = k(x)$ for k any field. Since $B = k[x]$ is a PID, it is Dedekind. Thus, for any prime $\mathcal{P}$ of B, the localization $B_{\mathcal{P}}$ is a DVR. Hence, for any irreducible polynomial $P \in k[x]$, we have a discrete valuation v_P on L, where $v_P(Q)$ is the highest power of P dividing Q (which is taken to be ∞ when $Q = 0$) for $Q \in B$ and $v_P(Q/R) = v_P(Q) - v_P(R)$ for $Q, R \in B$ and $R \neq 0$.

The product formula.

Suppose that we normalize the p-adic absolute values; that is, we set $\|x\|_p = p^{-v_p(x)}$. Then for any x, we have

$$\prod_p \|x\|_p = \frac{1}{p_1^{e_1} \cdots p_m^{e_m}}$$

where $x = \pm p_1^{e_1} \cdots p_m^{e_m}$. Let $\|x\|_\infty$ denote the usual absolute value $|x|$. Let

$$M_{\mathbb{Q}} = \{\text{ primes } p\} \cup \infty.$$

Then

$$\prod_{v \in M_{\mathbb{Q}}} \|x\|_p = 1.$$

This is called the product formula.

Similarly, working over $K[x]$, we call $P \in K[X]$ if P is monic, irreducible, and has degree greater than 0. We let $\|x\|_P = e^{-v_P(x)(\deg P)}$.

Then letting $\|x\|_\infty = e^{\deg x}$ (this measures the degree of the poll that x has at infinity), ane letting

$$M_{K[x]} = \{ \text{ primes } P \in K[X]\} \cup \infty,$$

we have

$$\prod_{v \in M_{K[x]}} \|x\|_v = 1.$$

Let K be a field with a discrete valuation v and let $\hat{K}$ be the completion of K with respect to $|\cdot|_v$ where

$$|x|_v = e^{-v(x)}.$$

We define B_v to be the set of $x \in \hat{K}$ with $v(x) \geq 0$ and let $\mathcal{M}_v$ be the maximal ideal in B_v. We see below that B_v is indeed a DVR.

Proposition 40.3. *With notation as above, v extends to a discrete valuation on $\hat{K}$.*

Proof. We take $v(x) = -\log \lim |x_i|$ for $x \neq 0$ represented by $(x_i)_{i=1}^\infty$. To see that this actually gives an integer, write $\lim |x_i| = C$ and if $-\log C$ is not an integer, we can pick pick ϵ so that $C - e^{-m} > \epsilon$ for all integers m. Then for any x_i, we have $|x_i| - \lim_i |x_i| > \epsilon$, which is impossible. Checking that $v(x)$ is multiplicative and $v(x+y) \leq \max(v(x), v(y))$ is simple. $\square$