Math 531 Tom Tucker
NOTES FROM CLASS 11/19

Next, we will show that $\ell(\mathcal{O}_L^*)$ is a sublattice in $\mathbb{R}^{r+s}$. We define a sublattice is a subgroup of $\mathbb{R}^m$ that has $\mathbb{Z}$-rank equal to the $\mathbb{R}$-dimension of the vector space it generates.

We were in the middle of proving the following.

Proposition 33.1. *Let $\mathcal{L}$ be a finitely generated subgroup of $\mathbb{R}^m$. Then $\mathcal{L}$ is a sublattice if and only if every bounded region in $\mathbb{R}^m$ contains at most finitely many elements of $\mathcal{L}$.*

Proof. Note, we already proved the "only if" part last week during our proof of the finiteness of the class group.

We will prove the "if" part by induction on m. If $m = 1$ and $\mathcal{L} \neq 0$ (0 is trivially a sublattice), then $\mathbb{R}^m = \mathbb{R}$, and we choose u to be the smallest positive number in $\mathcal{L}$. Then, for any $v \in \mathcal{L}$, we can write $v = tu + z$ where t is an integer and $0 \leq z < u$. But, since $z = v - tu$, we must have $z \in \mathcal{L}$, which means that $z = 0$ by the minimality of u. Thus, u must generate $\mathcal{L}$ as a $\mathbb{Z}$-module, so the rank of $\mathcal{L}$ as a group is equal to 1.

Now, we do the inductive step. Note that we may assume $\mathcal{L}$ generates $\mathbb{R}^m$ as a vector space, since otherwise it is contained in a vector space of dimension $\mathbb{R}^{m-1}$ and we are done by the inductive hypothesis. Thus, we can choose $\mathbb{R}$-linearly independent elements $v_1, \ldots, v_m$ of $\mathcal{L}$. By the inductive hypothesis, if V_0 is the $\mathbb{R}$-vector space generated by $v_1, \ldots, v_{m-1}$, then $\mathcal{L}_0 := V_0 \cap \mathcal{L}$ is a sublattice, and is a full lattice in V_0. Let $w_1, \ldots w_{m-1}$ be a basis for $\mathcal{L}_0$ (as a $\mathbb{Z}$-module). Then, $w_1, \ldots, w_{m-1}, v_m$ is a basis for $\mathbb{R}^m$, so any element of $\lambda \in \mathcal{L}$ can be written as

$$\lambda = \sum_{i=1}^{m-1} r_i w_i + r_m v_m$$

for real numbers r_i. Note that if $r_m = 0$, then $\lambda \in \mathcal{L}_0$, and we can choose all of the r_i to be integers. Note also that by subtracting off an appropriate element of $\mathcal{L}_0$, we obtain such a λ with all $0 \leq r_i < 1$ for $i \leq (m-1)$. There are only finitely many such λ with r_m also smaller than a certain bound (since any bounded region in $\mathbb{R}^m$ intersects $\mathcal{L}$ in finitely many points). Thus, there is a nonzero element λ' with $0 \leq r_i < 1$, for $i = 1, \ldots, m-1$ and $r_m > 0$ minimal (if $r_m = 0$, then the other r_i must be integers, we recall). I claim that $w_1, \ldots, w_{m-1}, \lambda'$ must be a $\mathbb{Z}$-basis for $\mathcal{L}$. Indeed, if we pick any element $\eta \in \mathcal{L}$ and write

$$\eta = \sum_{i=1}^{m-1} a_i w_i + a_m v_m$$

with $a_i \in \mathbb{R}$. Then by writing

$$a_m = tr_m + z$$

with $t \in \mathbb{Z}$ and $0 \leq z < r_m$ and subtracting

$$\sum_{i=1}^{m-1} ([a_i - r_i t]) w_i + t\lambda'$$

from η we obtain an element of $\mathcal{L}$ written as

$$\sum_{i=1}^{m-1} ((a_i - r_i t) - [a_i - r_i t]) w_i + z v_m$$

with $0 \leq z < a_m$. Thus, we must have $z = 0$ and

$$\eta - t\lambda' \in \mathcal{L}_O$$

and we are done. □

Let's define some notation now. For a finitely generated abelian group G we define $\mathrm{rk}(G)$ to be the free rank of G. Let's also define H to be the hyperplane $x_1 + \ldots x_{s+r} = 0$ in $\mathbb{R}^{s+r}$.

Proposition 33.2. *$\ell(\mathcal{O}_L^*)$ is a sublattice in H.*

Proof. Any bounded region in $\mathbb{R}^{s+r}$ is contained in a set Y_C consisting of all $(x_1, \ldots, x_{r+s})$ with $|x_i| \leq C$ for $C \geq 0$. For $b \in \mathcal{O}_L^*$, the absolute value of the i-th coordinate of $\ell(b)$ is less than or equal to C only if $|\sigma_i(b)| \leq e^C$ for all i. There are only finitely many such b by a Lemma from last time. □

Corollary 33.3.

$$\mathrm{rk}(\mathcal{O}_L^*) \leq (r + s - 1)$$

Proof. Since the kernel of ℓ is finite,

$$\mathrm{rk}(\mathcal{O}_L^*) = \mathrm{rk}(\ell(\mathcal{O}_L^*)).$$

From the previous Proposition we know that $\ell(\mathcal{O}_L^*)$ is sublattice in a vector space of dimension $s + r - 1$, so it must have $\mathbb{Z}$-rank at most $s + r - 1$. □

We're going to want use another embedding of $\mathcal{O}_L$ into an $\mathbb{R}$-vector space. This embedding, which we denote as h^* is almost exactly like the embedding h that we used earlier. It is

$$h^*(b) = (\sigma_1(b), \ldots, \sigma_r(b), \sigma_{r+1}(b), \ldots, \sigma_{r+s}(b)).$$

Note that is very similar to the embedding h used earlier. In fact, we can choose the $\mathbb{R}$-basis $x_1, \ldots, x_r$, $y_1, z_1, \ldots, y_s, z_1, \ldots, z_s$, where x_j is the element with j-th coordinate equal to 1 and all other coordinates equal to 0, y_j to be the the element with $(r+j)$-th element equal to 1 and all other coordinates equal to 0, and z_j to be the the element with $(r+j)$-th element equal to i and all other coordinates equal to 0. Then h is exactly the same with respect to its usual basis for V as h^* is with respect to the basis

$$x_1, \ldots, x_r, y_1, \ldots, y_s, z_1, \ldots, z_s.$$

If we give $\mathbb{R}^r \times \mathbb{C}^s$ the volume form associated to this basis, then

$$\mathrm{Vol}(h^*(\mathcal{O}_L)) = \mathrm{Vol}(h(\mathcal{O}_L)) = 2^{-s}\sqrt{\Delta(L/K)}.$$

In particular, $h^*(\mathcal{O}_L)$ is a full lattice in $\mathbb{R}^r \times \mathbb{C}^s$ (if it had $\mathbb{R}$-rank less than n, the volume would be 0).

The advantage of working with h^* is that ℓ is that if we denote as p_j projection onto the j-th coordinate (for $\mathbb{R}^r \times \mathbb{C}^s$). then

$$p_j(\ell(b)) = \log |p_j(h^*(b))|$$

for $1 \leq j \leq r$ and

$$p_j(\ell(b)) = 2\log |p_j(h^*(b))|$$

for $r+1 \leq j \leq r+s$.

We have already established that $h^*(\mathcal{O}_L)$ is a lattice so we should be able to find elements in it with certain properties. The idea roughly is this: we want to find a family of units u_i in $h^*(\mathcal{O}_L)$ for which we can control the $\pm$ sign of $\log |p_j(h^*(b))|$ for various j. We might hope that these units are linearly independent.

We will work with a region somewhat similar to the region we worked on when we were doing the finiteness of the class group. We define the region as follows. Let (t) be an $(r+s)$-tuple of positive numbers indexed as $(t)_i$. We define

$$\begin{aligned} Z_{(t)} :=& \{(x_1, \ldots, x_{s+r}) \in \mathbb{R}^r \times \mathbb{C}^s \mid |x_i| \leq (t)_i, 1 \leq i \leq r \\ & \text{and } |x_i|^2 \leq (t)_i \text{ for } r+1 \leq i \leq r+s\} \end{aligned}$$

The region $Z_{(t)}$ is just a cross product of regions in $\mathbb{R}$ and $\mathbb{C}$, specifically it is

$$[-(t)_1, (t)_1] \times \cdots \times [-(t)_r, (t)_r]$$
$$\times \{(x,y) \mid x^2 + y^2 \leq (t)_{r+1}^2\} \times \cdots \times \{(x,y) \mid x^2 + y^2 \leq (t)_{r+s}\}.$$

Thus,

$$\mathrm{Vol}(Z_{(t)}) = 2^r \pi^s t_1 \cdots t_{r+s}$$