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NOTES FROM CLASS 10/20

We will want to work with norms of ideals in a bit. There is one more thing to prove about norms first. First a Lemma.

(stuff from p. 24)

Lemma 22.1. *Let L be a separable (not necessarily Galois) field extension of K of degree n, let M be the Galois closure of L over K, and let $G = \mathrm{Gal}(M/L)$. Let H_L be the subgroup of G that acts trivially on L and let $H\backslash G$ be a complete set of left coset representatives for G over H. Then, for any $y \in L$, we have*

$$T_{L/K}(y) = \sum_{\sigma \in H\backslash G} \sigma(y)$$

and

$$\mathrm{N}_{L/K}(y) = \prod_{\sigma \in H\backslash G} \sigma(y)$$

Proof. Let $y_1, \ldots, y_m$. Then we know that

$$\mathrm{T}_{L/K}(y) = [L : K(y)] \left(\sum_{i=1}^{m} y_i \right)$$

and

$$\mathrm{N}_{L/K}(y) = \left(\prod_{i=1}^{m} y_i \right)^{[L:K(y)]}.$$

Now, let H_y be the subgroup of G that acts identially on $K(y)$. Then

$$\begin{aligned} \mathrm{T}_{L/K}(y) &= \sum_{\sigma \in H\backslash G} \sigma(y) = [L : K(y)] \sum_{\sigma \in H_y \backslash H} \sigma(y) \\ &= \mathrm{T}_{L/K}(y) = [L : K(y)] \left(\sum_{i=1}^{m} y_i \right), \end{aligned}$$

and

$$\begin{aligned} \mathrm{N}_{L/K}(y) &= \prod_{\sigma \in H\backslash G} \sigma(y) = \prod_{\sigma \in H_y \backslash H} \sigma(y)^{[L:K(y)]} \\ &= \mathrm{N}_{L/K}(y) = \left(\prod_{i=1}^{m} y_i \right)^{[L:K(y)]}, \end{aligned}$$

as desired. $\square$

Proposition 22.2. *Let $K \subseteq E \subseteq L$ be finite seprable extension of K. Then, for any $y \in L$, we have*

$$\mathrm{N}_{L/K}(y) = \mathrm{N}_{E/K}(\mathrm{N}_{L/E}(y)).$$

Proof. Let M be a Galois extension of K that contains L and let $G = \mathrm{Gal}(M/K)$. Let H_E and H_L be the subgroups of G that act identically on E and L respectively. Note that H_E is the Galois group for M over E. Let $\tau_1, \ldots, \tau_s$ represent the cosets $H_E \backslash G$ and $\gamma_1, \ldots, \gamma_t$ represent the cosets $H_L \backslash H_E$, then the $\tau_i \gamma_j$ represent the cosets $H_L \backslash G$. Therefore,

$$\mathrm{N}_{L/K}(y) = \prod_{i,j} (\tau_i \gamma_j)(y) = \prod_{i=1}^{s} \tau_i (\prod_{j=1}^{t} \gamma_j(y)) = \mathrm{N}_{E/K}(\mathrm{N}_{L/E}(y)).$$

□

One more thing to prove before getting to norms of ideals.

Proposition 22.3. *Let B be a Dedekind domain with finitely many maximal ideals $\mathcal{P}$. Then B is a principal ideal domain.*

Proof. It will suffice to show that every maximal ideal $\mathcal{P}$ of B is principal. Let $\mathcal{P}$ be a maximal ideal of B and let $\mathcal{Q}_1, \ldots, \mathcal{Q}_m$ be the other maximal ideals of B and let

$$I = \mathcal{Q}_1 \cdots \mathcal{Q}_m.$$

Then $\mathcal{P}^2 + I = 1$, so we can write $x + y = 1$ with $x \in \mathcal{P}^2$ and $y \in I$. Since $\mathcal{P} \neq \mathcal{P}^2$ (by unique factorization), there is some $a \in \mathcal{P} \setminus \mathcal{P}^2$. Let $\pi = ay + x$. Since

$$y = 1 - x \equiv 1 \pmod{\mathcal{P}^2},$$

we see that

$$ay + x \equiv ay \pmod{\mathcal{P}^2} \not\equiv 0 \pmod{\mathcal{P}^2},$$

so $ay \in \mathcal{P} \setminus \mathcal{P}^2$. Also

$$ay + x \equiv x \pmod{I} \equiv 1 - y \pmod{I} \equiv 1 \pmod{I},$$

so $ay + x \notin \mathcal{Q}_i$ for any i. Therefore $B\pi$ must be $\mathcal{P}$. □

Norms of ideals. Back on our usual set-up A Dedekind with field of fractions K, L a finite seprable extension of K of degree n, B the integral closure of A in L. We'll also want $A/\mathcal{P}$ to be perfect for every maximal ideal $\mathcal{P}$. We have already defined the norm $\mathrm{N}_{L/K} : L \longrightarrow K$; it sends B to A (since all the coefficients of the minimal polynomial of an integral element are integral). When it is clear what field we are working over we will omit the L/K subscript.

Definition 22.4. For any ideal $I \subset B$, we define the ideal $\mathrm{N}(I)$ to be the A-ideal generated by all $\mathrm{N}(x)$ for $x \in I$.

Properties of the norm (8.1 on p. 42)

Proposition 22.5. *The norm map has the following properties*

(1) $\mathrm{N}(By) = A\,\mathrm{N}(y)$ *for any* $y \in B$.
(2) *If* $S \subset A$ *is a multiplicative subset not containing 0, and* I *is an ideal of* B*, then* $\mathrm{N}(S^{-1}BI) = S^{-1}A\,\mathrm{N}(I)$.
(3) $\mathrm{N}(IJ) = \mathrm{N}(I)\,\mathrm{N}(J)$*, for any ideals* I *and* J *of* B.

Proof. 1. We know the norm map is multiplicative since the determinant of matrices is. Since $\mathrm{N}(B) \subset A$, it follows that $\mathrm{N}(By) \subset A\,\mathrm{N}(y)$. Also, $\mathrm{N}(y) \subset \mathrm{N}(By)$, so $A\,\mathrm{N}(y) \subset \mathrm{N}(By)$, so $\mathrm{N}(By) = A\,\mathrm{N}(y)$.

2. For any $y \in S^{-1}BI$, we can write $y = x/s$ for $x \in I$ and $s \in S$. Then $\mathrm{N}(y) = \mathrm{N}(x/s) = \mathrm{N}(x)/s^n \in S^{-1}A\,\mathrm{N}(I)$, so $\mathrm{N}(S^{-1}BI) \subseteq S^{-1}A\,\mathrm{N}(I)$. On the other hand, $S^{-1}A\,\mathrm{N}(I)$ is generated as an $S^{-1}A$-module by $\mathrm{N}(I)$ and $\mathrm{N}(I) \subseteq \mathrm{N}(S^{-1}BI)$, so we have $S^{-1}A\,\mathrm{N}(I) \subseteq \mathrm{N}(S^{-1}BI)$.

3. This is surprisingly difficult, since we the norm is not additive. On the other hand, since any ideal of A is determine by its localizations at all the maximal $\mathcal{P}$ of A, it will suffice to show that $A_{\mathcal{P}}\,\mathrm{N}(I)A_{\mathcal{P}}\,\mathrm{N}(J) = A_{\mathcal{P}}\,\mathrm{N}(IJ)$. From 2, this means we only have to show that

$$\mathrm{N}(S^{-1}BI)\,\mathrm{N}(S^{-1}BJ) = \mathrm{N}(S^{-1}BIJ).$$

Since there are finitely many primes $\mathcal{Q} \in B$ such that $\mathcal{Q} \cap A = \mathcal{P}$, the ring $S^{-1}B$ has finitely many primes, hence is a principal ideal domain. So we write $S^{-1}Bx = S^{-1}BI$ and $S^{-1}By = S^{-1}BJ$. Then we have

$$\begin{aligned}\mathrm{N}(S^{-1}BI)\,\mathrm{N}(S^{-1}BJ) &= \mathrm{N}(S^{-1}Bx)\,\mathrm{N}(S^{-1}By)\\ &= \mathrm{N}(S^{-1}Bxy) = \mathrm{N}(S^{-1}BIJ),\end{aligned}$$

and we are done. □