Math 531 Tom Tucker
NOTES FROM CLASS 10/13

We were in the middle of proving the following...

Lemma 19.1. *Let A and B' be as last time. Let $\mathcal{P}$ be a maximal prime of A, let $k = A/\mathcal{P}$, let $S = A \backslash \mathcal{P}$, and let $\phi : S^{-1}B' \longrightarrow S^{-1}B'/S^{-1}B'\mathcal{P}$ be the usual quotient map. Let us denote $S^{-1}B'/S^{-1}B'\mathcal{P}$ as C. Then for any $y \in S^{-1}B'$, we have $\phi(T_{L/K}(y)) = \mathrm{T}_{C/k}(\phi(y))$.*

Proof. Let $\bar{w}_1, \ldots, \bar{w}_n$ be a basis for C over k and pick $w_i \in B'$ such that $\phi(w_i) = \bar{w}_i$. Since the $\bar{w}_i$ are linearly independent, the w_i must be as well. To see this, suppose that $\sum_{i=1}^{n} a_i w_i = 0$ for $a_i \in S^{-1}B'$ (remember that everything in L is x/a for $x \in B'$ and $a \in A$). By dividing through by a power of a generator π for $A_{\mathcal{P}}\mathcal{P}$, we can assume that not all of the a_i are in $S^{-1}B'\mathcal{P}$. This means then that $\sum_{i=1}^{n} \phi(a_i)\bar{w}_i = 0$, with some $\phi(a_i) \neq 0$, which is impossible. Now, we are essentially done, since we can define the trace of any $y \in B'$ with respect to this basis. We have

$$y w_i = \sum_{j=1}^{n} m_{ij} w_j$$

with $m_{ij} \in A$, and

$$\phi(y)\bar{w}_i = \sum_{j=1}^{n} \phi(m_{ij})\bar{w}_j.$$

Hence,

$$\phi(\mathrm{T}_{L/K}(y)) = \sum_{i=1}^{n} \phi(m_{ii}) = \mathrm{T}_{C/k}(\phi(y)).$$

□

We need one quick lemma from linear algebra.

Lemma 19.2. *Let V be a vector space. Let $\phi : V \longrightarrow V$ be a linear map. Suppose that $\phi^k = 0$ for some $k \geq 1$. Then the trace of ϕ is zero.*

Proof. This is on your HW. □

When B is the integral closure of A in L, and $\mathcal{P}$ is maximal in A, we can write

$$\mathcal{P}B = \mathcal{Q}_1^{e_1} \cdots \mathcal{Q}_m^{e_m}.$$

If $e_i > 1$ for some i, then we say that $\mathcal{P}$ *ramifies* in B. When $B = A[\alpha]$, we know that $\mathcal{P}$ ramifies in B if and only if $\Delta(B/A) \subseteq \mathcal{P}$. That is true more generally.

Theorem 19.3. *Let B be the integral closure of A in L and let $\mathcal{P}$ be maximal in A. Then $\mathcal{P}$ ramifies in B if and only if $\Delta(B/A) \subseteq \mathcal{P}$.*

Proof. It will suffice to prove this locally, that is to say, it will suffice to replace A with $A_{\mathcal{P}}$ and B with $S^{-1}B$ where $S = A \setminus \mathcal{P}$. As in the previous Lemma, we write $k = A/\mathcal{P}$ and $C = S^{-1}B/\mathcal{P}S^{-1}B$ and let

$$\phi : S^{-1}B \longrightarrow S^{-1}B/\mathcal{P}S^{-1}B$$

Also, as in that Lemma let $\bar{w}_1, \ldots, \bar{w}_n$ be basis for C over k and pick $w_i \in S^{-1}B$ such that $\phi(w_i) = \bar{w}_i$. It is clear then that

$$A_{\mathcal{P}}w_1 + \ldots A_{\mathcal{P}}w_n + \mathcal{P}S^{-1}B = S^{-1}B,$$

so by Nakayama's Lemma, the w_i generate $S^{-1}B$ as an $A_{\mathcal{P}}$ module. From the Lemma above we have $T_{L/K}(w_iw_j) = T_{C/k}(\bar{w}_i\bar{w}_j)$, so the matrix $M = [\mathrm{T}_{C/k}(\bar{w}_i\bar{w}_j)]$ represents the form $(x, y) = \mathrm{T}_{C/k}(xy)$ on C/k. Let us now decompose C/k as ring, we have

$$C \cong S^{-1}B/\mathcal{P}S^{-1}B \cong \bigoplus_{i=1}^{m} S^{-1}B/S^{-1}B\mathcal{Q}_i^{e_i}$$

where

$$\mathcal{P}B = \mathcal{Q}_1^{e_1} \cdots \mathcal{Q}_m^{e_m}.$$

If $e_i > 1$, then any element $z \in C$ such that $z = 0$ in every coordinate but i and has i-th coordinate in $\mathcal{Q}_i$, has the property that $z^{e_i} = 0$. This means that the pairing

$$(x, y) = T_{C/k}(xy)$$

on C is degenerate from your homework.

If $e_i = 1$ for every i, then

$$C \cong S^{-1}B/S^{-1}B\mathcal{Q}_1 \oplus \cdots \oplus S^{-1}B/S^{-1}B\mathcal{Q}_m$$

and $S^{-1}B/S^{-1}B\mathcal{Q}_i$ is separable over k for each i. The trace form $(x, y) = \mathrm{T}_{C/k}(xy)$ decomposes into a sum of forms

$$(a, b) = \mathrm{T}_{(S^{-1}B/S^{-1}B\mathcal{Q}_i)/k}(ab),$$

each of which is nondegenerate, so (x, y) is nondegenerate, so

$$\det[\mathrm{T}_{L/K}(w_iw_j)] \notin \mathcal{P},$$

and we are done.

$\square$

Here is a simple and easy to prove fact comparing the discriminants of different subrings B and B' of L

Proposition 19.4. *Let $B' \subset B$ where B and B' are as usual (we will usually take B to the be the integral closure of A in L). Suppose that B has a basis $v_1, \ldots, v_n$ as an A-module and that B' has a basis $w_1, \ldots, w_n$ as an A-module. Writing*

$$w_i = \sum_{\ell=1}^{n} n_{i\ell} a_\ell,$$

and letting N be the matrix $[n_{i\ell}]$, we have

$$\det[\mathrm{T}_{L/K}(w_i w_j)] = (\det N)^2 \det[\mathrm{T}_{L/K}(v_i v_j)]. \tag{1}$$

Proof. Now,

$$\mathrm{T}_{L/K}(w_i w_j) = \sum_{\ell=1}^{n} \sum_{k=1}^{n} n_{i\ell} n_{jk} \, \mathrm{T}_{L/K}(v_i v_j).$$

A bit of linear algebra shows that this is exactly the same as the ij-th coordinate of the matrix $N^t M N$ where $M = [\mathrm{T}_{L/K}(v_i v_j)]$. Equation 1 follows. I gave an easier explanation on the board. $\square$