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NOTES FROM CLASS 10/11/04

For an element $\alpha \notin A$ that is integral over A, we define the discriminant $\Delta(\alpha/A)$ to be $\Delta(F)$ where F is the minimal monic for α over A. We also define the discriminant $\Delta(A[\alpha])$ to be $\Delta(A[\alpha])$.

Given a Dedekind domain A with field of fractions K and a finite separable extension L of K of degree n we want to be able to define a discriminant $\Delta(B'/A)$ of *any* subring B' of L. This will involve working with a basis for L over K that consists entirely of elements contained in B'

A bit more on subrings of the integral closure.

Proposition 18.1. *Let A be an integral domain with field of fractions K and let L be a finite extension of K. Suppose that $B' \subset L$ has field of fractions L and is integral over A. Then, for every element $y \in L$ there exists $a \in A$ such that $ay \in L$.*

Proof. Let $y = \alpha/\beta$ for $\alpha, \beta \in B'$ with $\alpha, \beta \neq 0$. We will show that $\alpha/\beta = b/a$ for $b \in B'$ and $a \in A$. We know that the ideal $B'\beta$ has nonzero intersection with A by taking the constant term of the minimal monic polynomial for β over A. Thus, we can write $\gamma\beta = a$ for some nonzero $a \in A$. Then $1/\beta = \gamma/a$, so $\alpha/\beta = \alpha\gamma/a$ and we are done, since this means that $a(\alpha/\beta) \in B'$. □

For the rest of class, A is Dedekind with field of fractions K, the field L is a finite separable extension of K of degree n, and B' is a subring of L that is integral over A. We will also assume that for every maximal ideal $\mathcal{P}$ of A, the residue field $A/\mathcal{P}$ is perfect.

We'll begin with a definition that works when B' is a free A-module, i.e. when B' is isomorphic as an A-module to A^n, where $n = [L : K]$. In this case, we choose a basis $w_1, \ldots, w_n$ for B' over A and we let M be the matrix $[m_{ij}]$ where $m_{ij} = \mathrm{T}_{L/K}(w_i w_j)$. Then we define

$$\Delta(B') = \det M. \tag{1}$$

How do we know that this agrees with our earlier definition in the case $B' = A[\alpha]$? In fact, it more or less follows from some earlier work we did. Recall that in this case, we can choose the basis $1, \alpha, \ldots, \alpha^{n-1}$, so that $[m_{ij}] = [\mathrm{T}_{L/K}(\alpha^{i+j-2})]$, which we recall is equal to

$$\sum_{\ell=1}^{n} \alpha_\ell^{i+j-2}.$$

As we saw earlier, letting N be the van der Monde matrix

$$\begin{pmatrix} 1 & \cdots & 1 \\ \alpha_1 & \cdots & \alpha_n \\ \cdots & \cdots & \cdots \\ \alpha_1^{n-1} & \cdots & \alpha_n^{n-1} \end{pmatrix},$$

we have $NN^t = M$, so

$$\det M = (\det N)^2 = \prod_{i<j} (\alpha_i - \alpha_j)^2,$$

which is the same as $\Delta(\alpha)$, so our definitions agree.

Not all B' will be free A-modules, however, so we have the more general definition below.

Definition 18.2. With notation as above $\Delta(B'/A)$ is defined to be ideal generated by the determinants of all matrices $M = [\mathrm{T}_{L/K}(w_i w_j)]$ as $w_1, \ldots, w_n$ range over all bases for L consisting of elements contained in B'.

Example 18.3. The reason that we need to talk about the discriminant relative to A is that B' could be defined over two different Dedekind domains. For example, we could take $B' = \mathbb{Z}[\sqrt{3}, \sqrt{7}]$ which is an extension of $\mathbb{Z}$ as well as of $\mathbb{Z}[\sqrt{3}]$ and $\mathbb{Z}[\sqrt{7}]$. The various discriminants $\Delta(B'/\mathbb{Z})$, $\Delta(B'/\mathbb{Z}[\sqrt{3}])$, and $\Delta(B'/\mathbb{Z}[\sqrt{7}])$ may all be different.

One nice fact about discriminants is that they can be computed locally. We have the following.

Proposition 18.4. *With notation as throughout lecture, let S be a multiplicative subset of A not containing 0. Then*

$$S^{-1}A\Delta(B'/A) = \Delta(S^{-1}B'/S^{-1}A).$$

Proof. Since any basis with elements in B' is also in $S^{-1}B'$, it is obvious that

$$S^{-1}A\Delta(B'/A) \subseteq \Delta(S^{-1}B'/S^{-1}A).$$

Similarly, given a basis $v_1, \ldots, v_n$ for L/K contained in $S^{-1}B'$, see that the basis $w_1, \ldots, w_n$ where $w_i = sv_i$ is contained in B' for some $s \in S$. Now

$$\det(T_{L/K}(w_i w_j)) = s^n \det(T_{L/K}(v_i v_j)),$$

so $S^{-1}A\Delta(B'/A) \supseteq \Delta(S^{-1}B'/S^{-1}A)$. $\square$

We know that $\Delta(B'/A)$ is an ideal I. If $I = \prod_{i=1}^{m} \mathcal{P}_i^{e_i}$, then $A_{\mathcal{P}_i} I = \mathcal{P}_i^{e_i}$, so to figure out what $\Delta(B'/A)$ is, all we have to do is figure out what $\Delta(S^{-1}B'/S^{-1}A)$ is for $S = A \setminus \mathcal{P}$.

The trace also behaves well with respect to reduction. Recall that as on the homework, whenever we have a finite integral extension of a field, we can define a trace. We'll apply that with the field $k = A/\mathcal{P}$ for a maximal ideal $\mathcal{P}$ of A. Since this computation is local, we will work over $A_{\mathcal{P}}$ (which is a DVR). This is just for simplicity, since we have $B'/\mathcal{P}B' \cong S^{-1}B'/S^{-1}B'\mathcal{P}$, so it isn't hard to see that the local computation gives the computation over A.

Lemma 18.5. *Let A and B' be as usual. Let $\mathcal{P}$ be a maximal prime of A, let $k = A/\mathcal{P}$, let $S = A \setminus \mathcal{P}$, and let $\phi : S^{-1}B' \longrightarrow S^{-1}B'/S^{-1}B'\mathcal{P}$ be the usual quotient map. Let us denote $S^{-1}B'/S^{-1}B'\mathcal{P}$ as C. Then for any $y \in S^{-1}B'$, we have $\phi(T_{L/K}(y)) = \mathrm{T}_{C/k}(\phi(y))$.*

Proof. Let $\bar{w}_1, \ldots, \bar{w}_n$ be a basis for C over k and pick $w_i \in B'$ such that $\phi(w_i) = \bar{w}_i$. Since the $\bar{w}_i$ are linearly independent, the w_i must be as well. To see this, suppose that $\sum_{i=1}^{n} a_i w_i = 0$ for $a_i \in S^{-1}B'$ (remember that everything in L is x/a for $x \in B'$ and $a \in A$). By dividing through by a power of a generator π for $A_{\mathcal{P}}\mathcal{P}$, we can assume that not all of the a_i are in $S^{-1}B'\mathcal{P}$. This means then that $\sum_{i=1}^{n} \phi(a_i)\bar{w}_i = 0$, with some $\phi(a_i) \neq 0$, which is impossible. Now, we are essentially done, since we can define the trace of any $y \in B'$ with respect to this basis. We have

$$yw_i = \sum_{j=1}^{n} m_{ij} w_j$$

with $m_{ij} \in A$, and

$$\phi(y)\bar{w}_i = \sum_{j=1}^{n} \phi(m_{ij})\bar{w}_j.$$

Hence,

$$\phi(\mathrm{T}_{L/K}(y)) = \sum_{i=1}^{n} \phi(m_{ii}) = \mathrm{T}_{C/k}(\phi(y)).$$

$\square$