Math 531
Notes from Class 10/01
Proposition 13.1. Let $A$ be a domain, $A \neq 0$, and let $B$ be integral over $A$. Then for any prime $\mathcal{P}$ of $A$, we have $B \mathcal{P} \neq 0$.

Proof. Suppose that $B \mathcal{P}=1$. Then there are $x_{1}, \ldots, x_{m} \in A$ such that

$$
b_{1} x_{1}+\cdots+b_{m} x_{m}=1
$$

Let $C=A\left[b_{1}, \ldots, b_{m}\right]$. Then $C$ is finitely generated as an $A$-module and $\mathbb{P} C=1$. Let $N=A_{\mathcal{P}} C$; then $N$ is finitely generated and $A_{\mathcal{P}} \mathcal{P} N=N$. Since $A_{\mathcal{P}}$ is local, we must have $N=0$ by Nakayama's lemma, which gives a contradiction, since $A \neq 0$.

Let's fix our notation for the rest of the day: $A$ is Dedekind with field of fractions $K, L \supseteq K$ is a finite separable field extension of degree n , and $B$ is the integral closure of $A$ in $L$. Sometimes, we will impose additional restrictions on $A$.

Corollary 13.2. If $A$ is a principal ideal domain and $[L: K]=n$ for $L$ a separable extension of $K$, the field of fractions of $A$, then the integral closure of $A$ in $L$ is isomorphic to $A^{n}$ as an $A$-module.

Proof. If $A$ is a principal ideal domain, then any finitely generated torsion-free $A$-module is a free module. In the proof of the theorem above, we saw that there is a free module of rank $n$, call it $M$ such that $M \subset B \subset M^{\dagger}$. Since $M^{\dagger}$ is also of rank $n$, we see that the rank of $B$ must be $n$.

One more thing I wanted to mention about factorizations of ideals in Dedekind domains. If $I \subseteq \mathcal{P}$, then $\mathcal{P}$ must appear in the factorization of $I$. This follows from the fact that $R_{\mathcal{P}} I$ is positive power of $R_{\mathcal{P}} \mathcal{P}$, which would not happen if $I$ didn't have $\mathcal{P}$ in its factorization.

Let us continue with the set-up: $A$ a Dedekind ring, $K$ field of fractions of $A, L$ a finite separable extension of $K$, and $B$ the integral closure of $A$ in $L$. We'll have $n=[L: K]$. Say we have a prime $\mathcal{P} \subset A$. What can we say about how $B \mathcal{P}$ factors?

Let's start with some basics. We write

$$
B \mathcal{P}=\mathcal{Q}_{1}^{e_{1}} \cdots \mathcal{Q}_{m}^{e_{m}} .
$$

The number $e_{i}$ is called the ramification degree of $\mathcal{Q}_{i}$ over $\mathcal{P}$. There's another number associated with $\mathcal{Q}_{i}$ over $\mathcal{P}$ as well. Recall that we have an injection of fields

$$
A / \mathcal{P} \hookrightarrow B / \mathcal{Q}_{i} .
$$

We call the index $\left[B / \mathcal{Q}_{i}: A / \mathcal{P}\right]$ the relative degree of $\mathcal{Q}_{i}$ over $\mathcal{P}$. It isn't hard to see that $f_{i}$ is finite and in fact $f_{i} \leq[L: K]$. We'll prove something more general along these lines in a bit. First, let's look at some examples...

Example 13.3. Let $A=\mathbb{Z}$ and $B=\mathbb{Z}[\sqrt{2}]$. Let's look at some factorizations of $B p$ into primes in $p$ for various $p$.
(1) $2 B=(\sqrt{2})^{2}$.
(2) $3 B$ is a prime.
(3) $7 B=(\sqrt{2}-3)(\sqrt{2}+3)$.

Theorem 13.4. With the set-up above, for $\mathcal{P}$ a maximal ideal of $A$ we have

$$
B \mathcal{P}=\mathcal{Q}_{1}^{e_{1}} \cdots \mathcal{Q}_{m}^{e_{m}}
$$

and $f_{i}=\left[B / \mathcal{Q}_{i}: A / \mathcal{P}\right]$ with

$$
\sum_{i=1}^{m} e_{i} f_{i}=n .
$$

Proof. We know that

$$
B / B \mathcal{P} \cong \sum_{i=1}^{m} B / \mathcal{Q}_{i}^{e_{i}}
$$

by the Chinese remainder theorem. Now, let $S=A \backslash \mathcal{P}$. Then from above, $S^{-1} B$ is the integral closure of $A_{\mathcal{P}}$ in $L$. Hence, it is isomorphic to $A_{\mathcal{P}}^{n}$ as an $A_{\mathcal{P}}$ module. It follows that $S^{-1} B / S^{-1} B \mathcal{P}$ is a $A_{\mathcal{P}} / \mathcal{P}$ vector space of dimension $n$. Moreover, since $S \cap \mathcal{Q}_{i}$ is empty for each $\mathcal{Q}_{i}$, we see that $S^{-1} B \mathcal{Q}_{i}$ is a prime in $S^{-1} B$ and we have

$$
S^{-1} B \mathcal{P}=S^{-1} B \mathcal{Q}_{1}^{e_{1}} \cdots \mathcal{Q}_{m}^{e_{m}}
$$

Combining this with homework results plus further localization, we obtain

$$
S^{-1} B / S^{-1} B \mathcal{P} \cong \sum_{i=1}^{m}\left(S^{-1} B\right) /\left(S^{-1} B \mathcal{Q}_{i}^{e_{i}}\right) \cong \sum_{i=1}^{m} B_{\mathcal{Q}_{i}} /\left(B_{\mathcal{Q}_{i}} \mathcal{Q}_{i}^{e_{i}}\right)
$$

Thus, we see that

$$
\operatorname{dim}_{A_{\mathcal{P}} / A_{\mathcal{P}} \mathcal{P}}\left(\sum_{i=1}^{m} B_{\mathcal{Q}_{i}} /\left(B_{\mathcal{Q}_{i}} \mathcal{Q}_{i}^{e_{i}}\right)\right)=n
$$

It will suffice to show, then, that

$$
\left.\operatorname{dim}_{( } A_{\mathcal{P}} / A_{\mathcal{P}} \mathcal{P}\right)\left(\sum_{i=1}^{m} B_{\mathcal{Q}_{i}} /\left(B_{\mathcal{Q}_{i}} \mathcal{Q}_{i}^{e_{i}}\right)\right)=\sum_{i=1}^{m} e_{i} f_{i},
$$

which would follow from

$$
\operatorname{dim}_{\left(A_{\mathcal{P}} / A_{\mathcal{P}} \mathcal{P}\right)}\left(B_{\mathcal{Q}_{i}} /\left(B_{\mathcal{Q}_{i}} \mathcal{Q}_{i}^{e_{i}}\right)\right)=e_{i} f_{i} .
$$

Since we can write
$0=B_{\mathcal{Q}_{i}} \mathcal{Q}_{i}^{e_{i}} /\left(B_{\mathcal{Q}_{i}} \mathcal{Q}_{i}^{e_{i}}\right) \subset\left(B_{\mathcal{Q}_{i}} \mathcal{Q}_{i}^{e_{i}}\right) /\left(B_{\mathcal{Q}_{i}}\right) \mathcal{Q}_{i}^{e_{i}-1} \subset \cdots \subset B_{\mathcal{Q}_{i}} /\left(B_{\mathcal{Q}_{i}} \mathcal{Q}_{i}^{e_{i}}\right)$, we need only show that

$$
\operatorname{dim}_{A_{\mathcal{P}} / \mathcal{P}}\left(\left(B_{\mathcal{Q}_{i}} \mathcal{Q}_{i}^{j}\right) /\left(B_{\mathcal{Q}_{i}} \mathcal{Q}_{i}^{j+1}\right)\right)=f_{i}
$$

for any $j \geq 0$. Note that since $B_{\mathcal{Q}_{i}}$ is a DVR, its its maximal ideal is generated by a single element $\pi$. It follows that each power $B_{\mathcal{Q}_{i}} \mathcal{Q}_{i}^{j}$ is generated by $\pi^{j}$ and that $\left(B_{\mathcal{Q}_{i}} \mathcal{Q}_{i}^{j}\right) /\left(B_{\mathcal{Q}_{i}} \mathcal{Q}_{i}^{j+1}\right)$ is therefore a 1-dimensional $B_{\mathcal{Q}_{i}} / B_{\mathcal{Q}_{i}} \mathcal{Q}_{i}$ vector space. Since $B / \mathcal{Q}_{i}$ is an $f_{i}$ dimensional $A / \mathcal{P}$-vector space, it follows that $\left(B_{\mathcal{Q}_{i}} \mathcal{Q}_{i}^{j}\right) /\left(B_{\mathcal{Q}_{i}} \mathcal{Q}_{i}^{j+1}\right)$ is an $f_{i}$-dimensional $A / \mathcal{P}$ vector space and we are done.

