Lecture Notes for Math 210 – 24 September 2007

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Chapter 3: Deterministic and Stochastic Calculus

§4.3 The integral

If f is a continuous function on $[0, T]$, then we define

$$\int_a^b f(x)\,dx \;=\; \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k\,,$$

where

$$a = x_0 \;<\; x_1 \;<\; \ldots \;<\; x_n = b\,.$$

and

$$\Delta x_k \;=\; x_k - x_{k-1}$$

x_k^* is any point such that $x_{k-1} \leq x_k^* \leq x_k$: your choice.

Example: $\int_a^b x\,dx$.

We set

$$x_k \;=\; a + \frac{k}{n}\,(b-a)\,.$$

That way $x_0 = a + \frac{0}{n}(b-a) = a$ and $x_n = a + \frac{n}{n}(b-a) = b$.

$$\Delta x_k \;=\; x_k - x_{k-1} \;=\; \left(a + \frac{k}{n}\,(b-a)\right) - \left(a + \frac{k-1}{n}\,(b-a)\right) \;=\; \frac{1}{n}\,(b-a)\,.$$

Let $x_k^* = x_k$.

Riemann sum:

$$\begin{aligned}
\sum_{k=1}^{n} f(x_k)\,\Delta x_k \;&=\; \sum_{k=1}^{n} x_k\,\Delta x_k \\
&=\; \sum_{k=1}^{n} \left(a + \frac{k}{n}\,(b-a)\right)\,\frac{1}{n}\,(b-a) \\
&=\; \sum_{k=1}^{n} a(b-a)\,\frac{1}{n} + \sum_{k=1}^{n} (b-a)^2\,\frac{k}{n^2} \\
&=\; a(b-a) + \frac{(b-a)^2}{n^2}\sum_{k=1}^{n} k\,.
\end{aligned}$$

The last equation is because $\sum_{k=1}^{n} c = cn$ for a constant c independent of k.

There is a useful formula

$$\sum_{k=1}^{n} k \;=\; \frac{n(n+1)}{2}\,.$$

So

$$a(b-a) + \frac{(b-a)^2}{n^2}\sum_{k=1}^{n} k \;=\; a(b-a) + \frac{(b-a)^2(n+1)}{2n}\,.$$

Now

$$
\begin{aligned}
\int_a^b x\,dx &= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x_k \\
&= \lim_{n\to\infty} \left[a(b-a) + \frac{(b-a)^2(n+1)}{2n} \right] \\
&= a(b-a) + \frac{(b-a)^2}{2} \\
&= \frac{b^2}{2} - \frac{a^2}{2}\,.
\end{aligned}
$$

We never really use the definition of the integral to calculate specific integrals.

But like the owner's manual for your car, the definition of the integral is useful when something goes wrong.

Something will go wrong a bit later (with Brownian motion), and we will have to refer to the definition.

§5.3 Taylor Series Expansion

Let $f(x)$ be an infinitely differentiable function of $x \in \mathbb{R}$ and pick and arbitrary value of x, called x_0.

Definition

The Taylor expansion of $f(x)$ around $x_0 \in \mathbb{R}$ is defined as

$$
\begin{aligned}
f(x) &= f(x_0) + (x-x_0)\,\frac{df}{dx}(x_0) + \frac{(x-x_0)^2}{2}\cdot\frac{d^2 f}{dx^2}(x_0) + \frac{(x-x_0)^3}{3!}\cdot\frac{d^3 f}{dx^3}(x_0) + \ldots \\
&= \sum_{n=0}^{\infty} \frac{(x-x_0)^n}{n!}\cdot\frac{d^n f}{dx^n}(x_0)\,.
\end{aligned}
$$

Example: $f(x) = e^x$. What is the Taylor expansion about $x = 0$?

We know $f'(x) = e^x = f(x)$.

That implies that $\frac{d^n f}{dx^2} = e^x$ for all $n \geq 0$.

Also, $e^0 = 1$.

So

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{(x-0)^n}{n!} \cdot \frac{d^n f}{dx^n}(0) \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} e^0 \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} . \end{aligned}$$

We are going to need the Taylor expansion as a purely theoretical tool when we consider the change-of-variables formula for stochastic integration.

Let us start by considering the change-of-variables formula for regular integration using the definition of the integral.

§4.3.2 The Stieltjes integral : aka change-of-variables

The result of this calculation is not that important or new: you learned ordinary change-of-variables a long time ago in calculus. It is the method you use whenever you solve an integral by substitution.

But our mathematical approach to the problem is important, because this is what we will do when we eventually do stochastic integration.

Suppose that $f(x)$ is an integrable function.

Suppose $x = g(y)$ for an infinitely differentiable, increasing function $g : \mathbb{R} \to [a, b]$, where $g(c) = a$ and $g(d) = b$.

$$\begin{aligned}\int_a^b f(x)\,dx &= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,[x_k - x_{k-1}] \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} f(g(y_k))\,[g(y_k) - g(y_{k-1})]\,,\end{aligned}$$

where $c = y_0 < y_1 < \cdots < y_n = d$ are such that $g(y_k) = x_k$.

By Taylor's expansion,

$$\begin{aligned} g(y_k) - g(y_{k-1}) &= g(y_k) - \left[g(y_k) + (y_{k-1} - y_k)\,g'(y_k) + \frac{(y_{k-1} - y_k)^2}{2}\,g''(y_k) + \ldots \right] \\ &= g(y_k) - \left[g(y_k) - g'(y_k)\,\Delta y_k + \frac{1}{2} g''(y_k), (\Delta y_k)^2 + \ldots \right] \\ &= g'(y_k)\,\Delta y_k - \frac{1}{2}\,g''(y_k)\,(\Delta y_k)^2 + \ldots \end{aligned}$$

Plugging into Riemann sum:

$$\begin{aligned} \int_a^b f(x)\,dx &= \lim_{n\to\infty} \sum_{k=1}^{n} f(g(y_k))\ \left[g'(y_k)\,\Delta y_k - \frac{1}{2}\,g''(y_k)\,(\Delta y_k)^2 + \ldots \right] \\ &= \lim_{n\to\infty} \left[\sum_{k=1}^{n} f(g(y_k))\,g'(y_k)\,\Delta y_k - \sum_{k=1}^{\infty} \frac{1}{2} f(g(y_k))\,g''(y_k)\,(\Delta y_k)^2 + \ldots \right] \end{aligned}$$

Now,

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(g(y_k))\,g'(y_k)\,\Delta y_k = \int_c^d f(g(y))\,g'(y)\,dy\,.$$

Q: Suppose $\Delta y_k = \Delta y$ for all k. What is $\lim_{n\to\infty} \Delta y$?

A: 0 because then $\Delta y = (d - c)/n \to 0$ as $n \to \infty$.

Q: What is $\lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{1}{2} f(g(y_k))\,g''(y_k)\,(\Delta y_k)^2$ equal to?

A: It is 0 because you have a squared-power of Δy_k in the summation. We know if we only had one, we would get

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{1}{2} f(g(y_k))\, g''(y_k)\, \Delta y_k \;=\; \int_c^d \frac{1}{2} f(g(y))\, g''(y)\, dy\,.$$

But since we have two, we use the product rule for limits.

Suppose $\Delta y_k = \Delta y$ for all k:

$$\begin{aligned}
\lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{1}{2} f(g(y_k))\, g''(y_k)\, (\Delta y_k)^2 \;&=\; \lim_{n\to\infty} (\Delta y) \sum_{k=1}^{n} \frac{1}{2} f(g(y_k))\, g''(y_k)\, (\Delta y_k)^2 \\
&= \left(\lim_{n\to\infty} \Delta y\right) \left(\lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{1}{2} f(g(y_k))\, g''(y_k)\, \Delta y_k\right) \\
&= 0 \cdot \int_c^d \frac{1}{2} f(g(y))\, g''(y)\, dy \\
&= 0\,.
\end{aligned}$$

Also, if you ever have higher power of Δy than 2, you also get 0.

So all the $\ldots$ in the Riemann sum also go to zero.

Conclusion:

$$\int_a^b f(x)\, dx \;=\; \int_c^d f(g(y))\, g'(y)\, dy\,,$$

if $g : [c, d] \to [a, b]$ is a strictly increasing, differentiable, one-to-one onto map.

Two points. First, we will abbreviate one of the previous facts as $(dy)^2 = 0$.

By this we mean the following.

dy is really Δy, in the Riemann sum, in the limit that $\Delta y \to 0$.

Even though $\Delta y \to 0$, the number of terms in the sum is n which is like $1/\Delta y$.

So the total Riemann sum does not go to zero.

That is why $\int f(y)\,dy$ is not always zero.

But, if you put $(dy)^2$, then it is like you are putting $(\Delta y)^2$ in the Riemann sum, and that does go to zero, because there are only $1/\Delta y$ terms.

$$\boxed{(dy)^2 = 0}$$

Similarly, $(dx)^2 = 0$, $(dt)^2 = 0$.

Second, if we did not have $(dy)^2 = 0$, just theoretically, we would have had to have kept more than just the first term in the Taylor series.

We would have also obtained at least the term like $\int f(g(y))\,g''(y)\,(dy)^2$.

Of course this is ridiculous because we do know $(dy)^2 = 0$.

But, when we cover Brownian motion, we are not going to have $(dB_t)^2 = 0$. Instead we will have $(dB_t)^2 = dt$.

In that case, we will definitely have to keep the second-order term from the Taylor expansion.

Of course, that is all in the future!