Lecture Notes for Math 210 – 21 September 2007

Shannon Starr

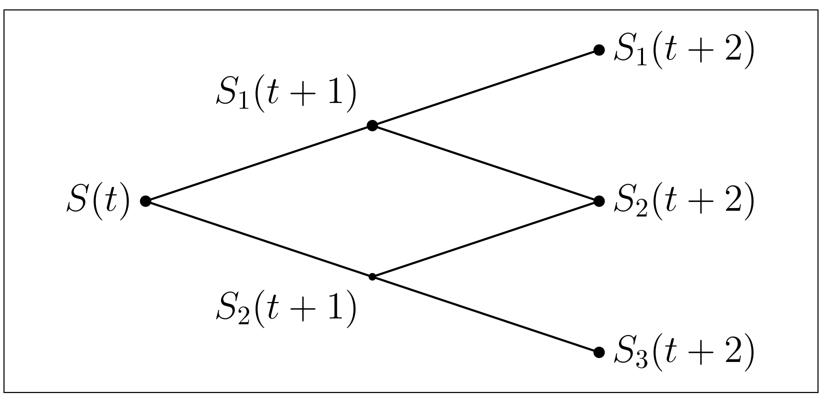
21 September 2007

Chapter 2: A Primer on the Arbitrage Theorem

Subtitle: The binomial tree model

Lattice models (aka binomial tree model)

Exercise Consider the following 2-step binomial tree with each step representing an extra time of $\Delta = 1$ year.



We have the following values

$$S(t) = \$100\,, \qquad \begin{bmatrix} S_1(t+1) \\ S_2(t+1) \end{bmatrix} = \begin{bmatrix} \$150 \\ \$66.6\bar{6} \end{bmatrix}, \qquad \begin{bmatrix} S_1(t+2) \\ S_2(t+2) \\ S_3(t+2) \end{bmatrix} = \begin{bmatrix} \$225 \\ \$100 \\ \$44.4\bar{4} \end{bmatrix}.$$

Suppose $r = 4.75\%$ (for both 1 year and for the period from 1 to 2 years).

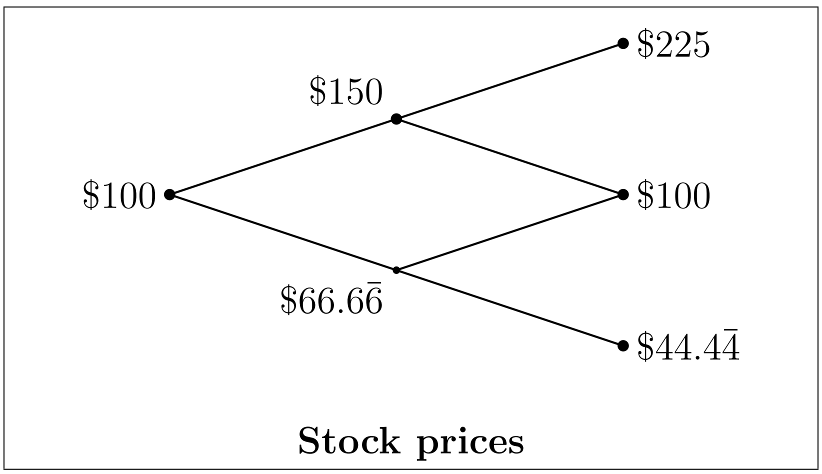
Consider a call option, whose expiration date is $T = t+2\Delta$ and suppose $K = \$100$.

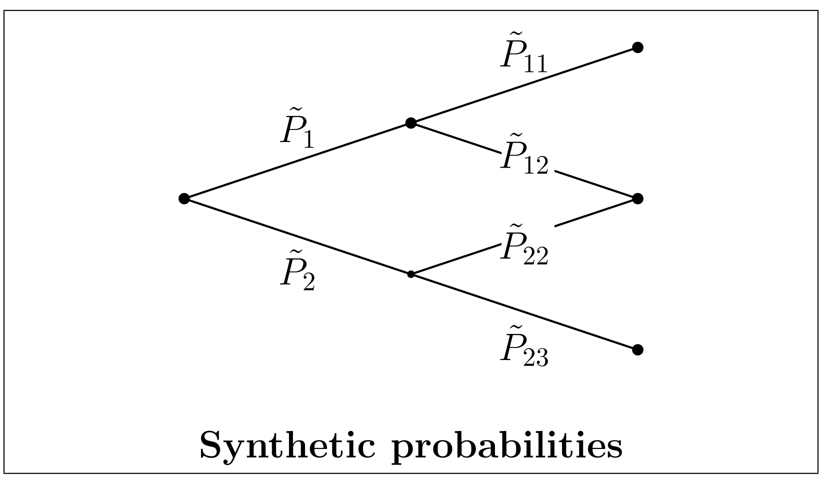
Find the payoffs at expiration for the call option, depending on $S(t+2\Delta)$.

Then find the possible values of the call option at time $t+\Delta$, depending on $S(t+\Delta)$.

Finally, find the correct price for the call option today $C(t)$.

Answer:

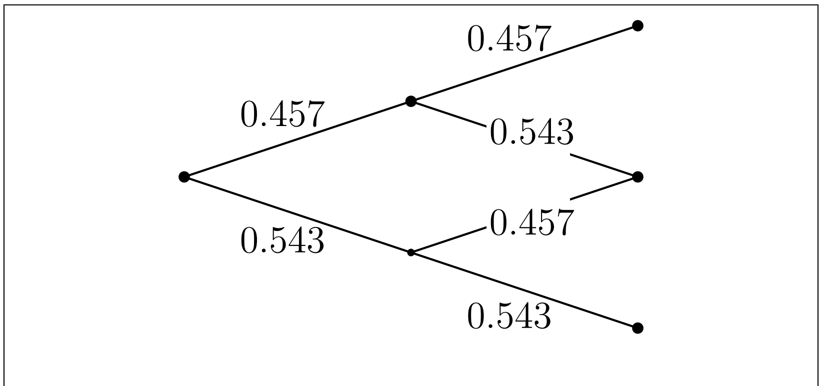




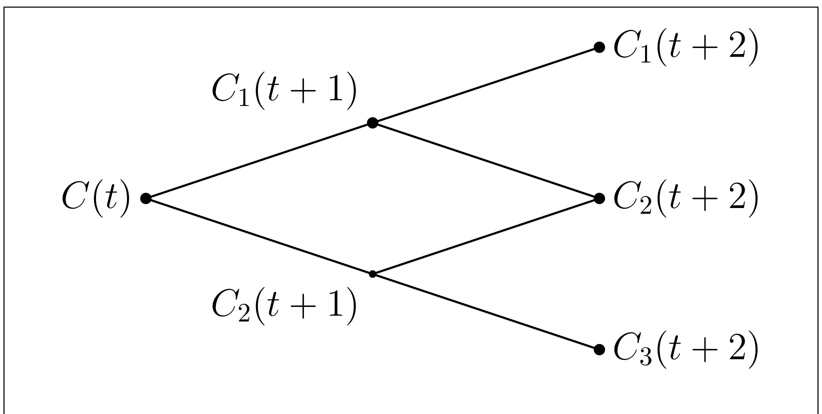
$$\tilde{P}_1 = \frac{(1+r\Delta)S(t) - S_2(t+\Delta)}{S_1(t+\Delta) - S_2(t+\Delta)} = \frac{(1.0475)\$100 - \$66.6\bar{6}}{\$150 - \$66.6\bar{6}} = \frac{\$38.08\bar{3}}{\$83.3\bar{3}} = 0.457$$

$$\tilde{P}_2 = 1 - \tilde{P}_1 = 0.543$$

Since the up-step is always 3/2 of the present value, and the down-step is always 2/3 of the present value, these will be the synthetic probabilities for all the steps:



Synthetic probabilities



Call option prices

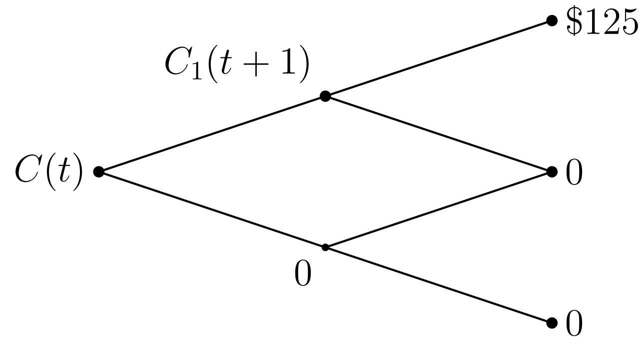
The payoff for the call option at time $T = t+2$ is

$$C(t+2) \;=\; \begin{cases} C_1(t+2) & \text{if } S(t+2) = S_1(t+2) \\ C_2(t+2) & \text{if } S(t+2) = S_2(t+2) \\ C_3(t+2) & \text{if } S(t+2) = S_3(t+2) \end{cases}$$

$$C_1(t+2) \;=\; \max(S_1(t+2) - K, 0) \;=\; \max(\$225 - \$100, 0) \;=\; \$125$$

$$C_2(t+2) \;=\; \max(S_2(t+2) - K, 0) \;=\; \max(\$100 - \$100, 0) \;=\; \$0$$

$$C_3(t+2) \;=\; \max(S_3(t+2) - K, 0) \;=\; \max(\$44.4\bar{4} - \$100, 0) \;=\; \$0$$



Call option prices

We know that $C_2(t+1) = 0$ since, if $S(t+1) = S_2(t+1)$, then no matter what happens in the next step, the call option will not be exercised.

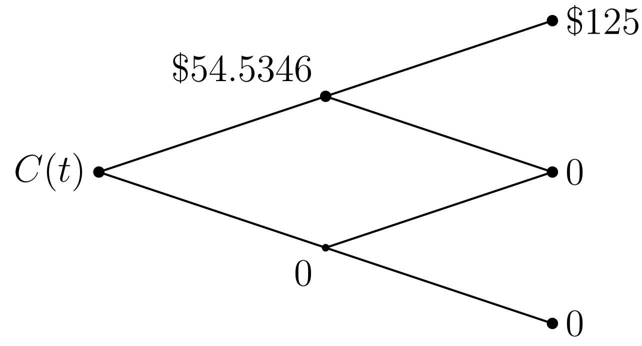
To find out $C_1(t+1)$, we use the formula:

$$C_1(t+1) \;=\; (1 + r\Delta)^{-1} E^{\tilde{P}}[C(t+2) \,|\, S(t+1) = S_1(t+1)]\,.$$

We mainly write the conditional probability for our own edification.

It just reminds us that we do not have to include the probability that $S(t+1)$ actually equals $S_1(t+1)$, just the two probabilities for the next step.

$$\begin{aligned} C_1(t+1) \;&=\; \frac{\tilde{P}_{11} \cdot C_1(t+2) + \tilde{P}_{12} \cdot C_2}{1.0475} \\ &=\; \frac{(0.457)\$125 + (0.543)0}{1.0475} \\ &=\; \frac{\$57.125}{1.0475} \\ &\approx \$54.5346 \end{aligned}$$



Call option prices

Finally, we use the formula

$$C(t) \ = \ (1 + r\Delta)^{-1} \, E^{\tilde{P}}[C(t+1)] \, .$$

(Now we do include the probabilities for the first step, and only those.)

$$\begin{aligned} C(t) \ &= \ \frac{\tilde{P}_1 \cdot C_1(t+1) + \tilde{P}_2 \cdot C_2}{1.0475} \\ &= \ \frac{(0.457)\$54.5346 + (0.543)0}{1.0475} \\ &= \ \frac{\$24.9223}{1.0475} \\ &\approx \$23.7922 \\ &\approx \$23.79 \end{aligned}$$

Chapter 3: Deterministic and Stochastic Calculus

This chapter has two purposes. The first goal is to remind *you of some things you learned in calculus, but may have forgotten. The second goal is to* introduce *you to some of the finer points of calculus that you probably have not covered before, unless you have already had a graduate course in analysis.*

§3.2 - 3.3 Some tools of standard calculus : Functions

Subtitle: *Some standard tools of calculus.*

A function $f : A \to B$ is an assignment of a point in B for every point in A.

A is called the domain.

B is called the range, or perhaps it is better to call B the target space.

Sometimes we refer to the set of values that f can take as the range:

$$\text{Ran}(f) \;=\; \{b \,:\, b = f(a) \text{ for some } a \in A\}\,.$$

These details are not that important right now.

§3.3.2 Examples of functions: The exponential and logarithm functions.

Recall that the number e is defined by

$$e \;=\; \sum_{n=0}^{\infty} \frac{1}{n!} \;\approx\; 2.71828\,.$$

It is easy to see that

$$\frac{d}{dx}e^x \;=\; e^x\,,$$

for all $x \in \mathbb{R}$.

The function $f(x) = e^x$ has domain equal to all of $\mathbb{R}$,

but its range is really only the positive numbers $(0, \infty)$.

Also, by properties of the exponent, it is obvious that

$$e^{x+y} \;=\; e^x \, e^y \,.$$

The ordinary differential equation (ODE)

$$f'(x) = f(x) \quad \text{which is the same as} \quad \frac{d}{dx} f(x) \;=\; f(x)\,,$$

is important.

One solution is $f(x) = e^x$.

We will find all solutions shortly.

The natural logarithm function is denote $\ln(x)$.

It is defined for x in the domain $(0, \infty)$, only.

(But its range is all of $\mathbb{R}$.)

It is defined as the inverse function to e^x:

$$y = \ln(x) \quad \text{if and only if } x \;=\; e^y \,.$$

I.e.,

$$\ln(e^y) \;=\; y \quad \text{and} \quad e^{\ln(x)} \;=\; x\,.$$

Recall the chain-rule for derivatives

$$\frac{d}{dx} f(g(x)) \;=\; f'(g(x))\, g'(x)\,.$$

Using this we see that

$$\frac{d}{dx} e^{\ln(x)} \;=\; \frac{d}{dy} e^{y}\bigg|_{y=\ln(x)} \frac{d}{dx} \ln(x) \;=\; e^{\ln(x)} \frac{d}{dx} \ln(x) \;=\; x \frac{d}{dx} \ln(x)\,.$$

But also, since $e^{\ln(x)} = x$ we know

$$\frac{d}{dx} e^{\ln(x)} \;=\; \frac{d}{dx}(x) \;=\; 1\,.$$

So

$$x \frac{d}{dx} \ln(x) \;=\; 1 \qquad \Longrightarrow \qquad \frac{d}{dx} \ln(x) \;=\; \frac{1}{x}\,.$$

§3.5.4 Ordinary differential equations

Even though this is out-of-order, it seems good to talk about ODE's here. Ordinary differential equations ODE's are algebraic equations involving a function, $f(x)$ and some of its derivatives $f'(x), f''(x), \ldots$.
An important ODE is the equation

$$f'(x) \;=\; r\, f(x)\,.$$

If we write $y = f(x)$, then this could be denoted as

$$\frac{dy}{dx} \;=\; r\, y\,.$$

The key to solving ODE's is usually to use the fundamental theorem of calculus (FTC):

Suppose that

$$f'(x) \;=\; r f(x)\,.$$

Let us divide both sides by $f(x)$ to get

$$\frac{f'(x)}{f(x)} \;=\; r\,.$$

Next, notice by the chain-rule that

$$\frac{d}{dx}\,\ln\big(f(x)\big) \;=\; \frac{d}{dy}\ln(y)\bigg|_{y=f(x)} f'(x) \;=\; \frac{1}{f(x)}\,f'(x)\,.$$

Therefore, the ODE is equivalent to

$$\frac{d}{dx}\,\ln\big(f(x)\big) \;=\; r\,.$$

Taking the antiderivative gives

$$\ln\big(f(x)\big) \;=\; rx + C\,,$$

which is equivalent to

$$f(x) \;=\; A\,e^{rx}\,.$$

If we rewrite the ODE as

$$\frac{d}{dt}B_t \;=\; r\,B_t\,,$$

which is the fundamental equation for a bond price, then we get

$$B_t \;=\; e^{rt}\,B_0\,.$$

Or, if we start at another time t_0 instead of 0,

$$B_t = e^{r(t-t_0)} B_{t_0} .$$

You derive an equation like this in your HW2 for problem 3.