Lecture Notes for Math 210 – 19 September 2007

Shannon Starr

19 September 2007

Chapter 2: A Primer on the Arbitrage Theorem

Subtitle: The binomial tree model

Theorem

In the binomial model, the principle of *no-arbitrage* implies that there are two constants $\psi_1, \psi_2 \geq 0$ such that

$$\begin{bmatrix} B(t) \\ S(t) \\ C(t) \end{bmatrix} = \begin{bmatrix} B(t+\Delta) & B(t+\Delta) \\ S_1(t+\Delta) & S_2(t+\Delta) \\ C_1(t+\Delta) & C_2(t+\Delta) \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \qquad (*)$$

Assuming $S_1(t+\Delta) \neq S_2(t+\Delta)$, the constants ψ_1 and ψ_2 are unique.

Synthetic probabilities

Define

$$\tilde{P}_1 \;=\; (1+r\Delta)\psi_1 \;=\; \frac{(1+r\Delta)S(t) - S_2(T)}{S_1(T) - S_2(T)}$$

and

$$\tilde{P}_2 \;=\; (1+r\Delta)\psi_2 \;=\; \frac{S_1(T) - (1+r\Delta)S(t)}{S_1(T) - S_2(T)}$$

Since ψ_1 and ψ_2 are nonnegative, the same is true of $\tilde{P}_1$ and $\tilde{P}_2$.

Also

$$\begin{aligned} \tilde{P}_1 + \tilde{P}_2 \;&=\; \frac{(1+r\Delta)S(t) - S_2(T)}{S_1(T) - S_2(T)} + \frac{S_1(T) - (1+r\Delta)S(t)}{S_1(T) - S_2(T)} \\ &=\; \frac{S_1(T) - S_2(T)}{S_1(T) - S_2(T)} \\ &=\; 1\,. \end{aligned}$$

<u>Recall "probabilities"</u>: A set of numbers assigned to possible outcomes of an unknown random variable are called *probabilities* if they are all nonnegative and add up to 1.

$\tilde{P}_1 =$ "synthetic probability" or "risk neutral probability" that $S(t+\Delta) = S_1(t+\Delta)$

$\tilde{P}_2 =$ "synthetic" or "risk neutral" probability that $S(t+\Delta) = S_2(t+\Delta)$

<u>∗ Note:</u> These are *not* the real probabilities for $S(t+\Delta)$, not even in this imaginary binomial universe. We will explain more, briefly.

Q: What is the definition of the "expectation"?

If a random variable X can take values $x_1, \ldots, x_n$, with probabilities $p_1, \ldots, p_n$, then we say the *expectation* of X (or its "expected value") is

$$E^p[X] := p_1 \cdot x_1 + \cdots + p_n \cdot c_n \,.$$

- The risk-neutral expected value of the bond:

$$E^{\tilde{P}}[B(t+\Delta)] = \tilde{P}_1 B(t+\Delta) + \tilde{P}_2 B(t+\Delta) = (1+r\Delta)\, B(t)\,.$$

- The risk-neutral expected value of the stock:

$$E^{\tilde{P}}[S(t+\Delta)] := \tilde{P}_1 S_1(t+\Delta) + \tilde{P}_2 S_2(t+\Delta) = (1+r\Delta)\, S(t)\,,$$

- The risk-neutral expected value of the call option:

$$E^{\tilde{P}}[C(t+\Delta)] := \tilde{P}_1 C_1(t+\Delta) + \tilde{P}_2 C_2(t+\Delta) = (1+r\Delta)\, C(t)\,,$$

These calculations are not trivial, but they follow from the theorem.

$\ast$ Note: The difference between real probabilities and synthetic probabilities is this. According to synthetic probabilities, the current price of an asset is the expected value of its future value, discounted to get its present value. But in the real world, investors have to also get a premium for buying a risky asset. The basic assumption is that, if an investor has the choice between receiving $50 today, or receiving a ticket that pays-off $100 with a 50% probability and $0 with a 50% probability, the investor will prefer the sure thing. Investors are *risk-averse*, whereas the synthetic probabilities are *risk-neutral*.

Lattice models (aka binomial tree model)

For the purpose of calculating $C(t)$ we can summarize as follows:

- First, calculate $\tilde{P}_1$ and $\tilde{P}_2$ as

$$\tilde{P}_1 \;=\; \frac{(1+r\Delta)S(t) - S_2(T)}{S_1(T) - S_2(T)} \quad \text{and} \quad \tilde{P}_2 \;=\; 1 - \tilde{P}_1$$

- Second, calculate $C_1(T)$ and $C_2(T)$ using the formulas

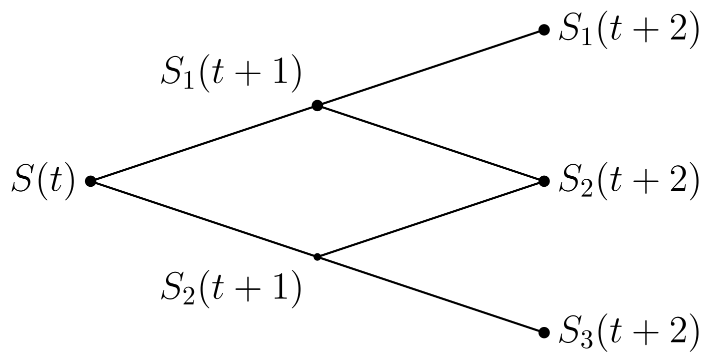
$$C_1(T) \;=\; \max(S_1(T) - K, 0) \quad \text{and} \quad C_2(T) \;=\; \max(S_2(T) - K, 0)\,.$$

- Third, calculate $E^{\tilde{P}}[C(T)]$ as

$$E^{\tilde{P}}[C(T)] \;=\; \tilde{P}_1 \cdot C_1(T) + \tilde{P}_2 \cdot C_2(T)\,.$$

- Finally, set $C(t) = (1+r\Delta)^{-1}\, E^{\tilde{P}}[C(T)]$.

Exercise Consider the following 2-step binomial tree with each step representing an extra time of $\Delta = 1$ year.



We have the following values

$$S(t) = \$100\,, \qquad \begin{bmatrix} S_1(t+1) \\ S_2(t+1) \end{bmatrix} = \begin{bmatrix} \$150 \\ \$66.6\bar{6} \end{bmatrix}, \qquad \begin{bmatrix} S_1(t+2) \\ S_2(t+2) \\ S_3(t+2) \end{bmatrix} = \begin{bmatrix} \$225 \\ \$100 \\ \$44.4\bar{4} \end{bmatrix}.$$

Suppose $r = 4.75\%$ (for both 1 year and for the period from 1 to 2 years).

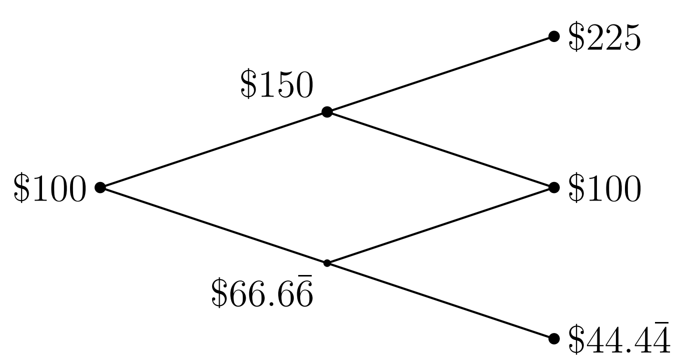
Consider a call option, whose expiration date is $T = t + 2\Delta$ and suppose $K = \$100$.

Find the payoffs at expiration for the call option, depending on $S(t + 2\Delta)$.

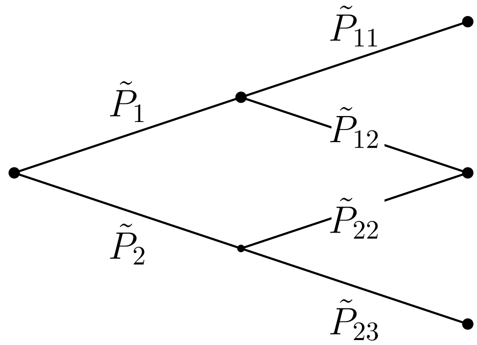
Then find the possible values of the call option at time $t + \Delta$, depending on $S(t + \Delta)$.

Finally, find the correct price for the call option today $C(t)$.

Answer:



Stock prices

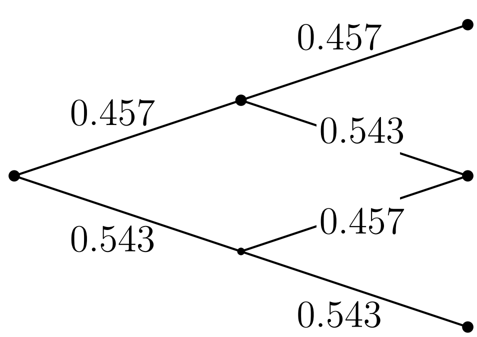


Synthetic probabilities

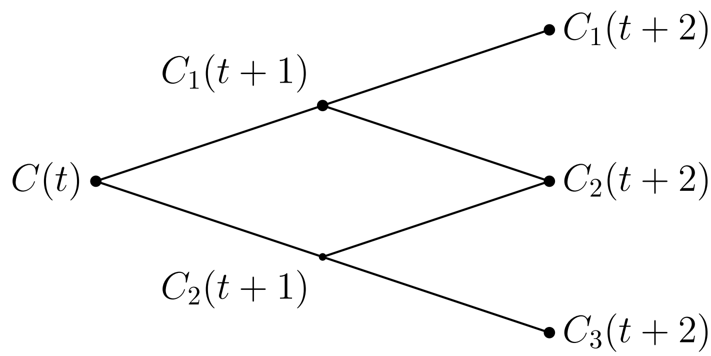
$$\tilde{P}_1 \;=\; \frac{(1+r\Delta)S(t) - S_2(t+\Delta)}{S_1(t+\Delta) - S_2(t+\Delta)} \;=\; \frac{(1.0475)\$100 - \$66.6\bar{6}}{\$150 - \$66.6\bar{6}} \;=\; \frac{\$38.08\bar{3}}{\$83.3\bar{3}} \;=\; 0.457$$

$$\tilde{P}_2 \;=\; 1 - \tilde{P}_1 \;=\; 0.543$$

Since the up-step is always 3/2 of the present value, and the down-step is always 2/3 of the present value, these will be the synthetic probabilities for all the steps:



Synthetic probabilities



Call option prices

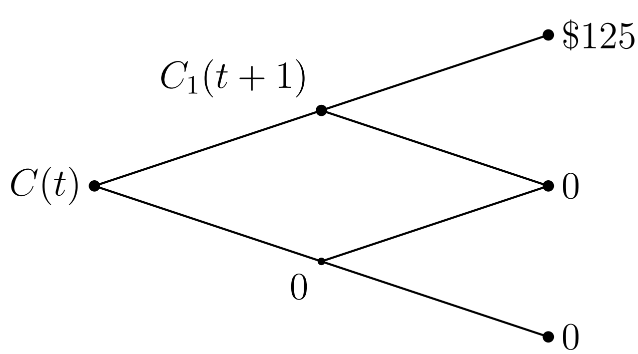
The payoff for the call option at time $T = t + 2$ is

$$C(t+2) \;=\; \begin{cases} C_1(t+2) & \text{if } S(t+2) = S_1(t+2) \\[2ex] C_2(t+2) & \text{if } S(t+2) = S_2(t+2) \\[2ex] C_3(t+2) & \text{if } S(t+2) = S_3(t+2) \end{cases}$$

$$C_1(t+2) \;=\; \max(S_1(t+2) - K, 0) \;=\; \max(\$225 - \$100, 0) \;=\; \$125$$

$$C_2(t+2) \;=\; \max(S_2(t+2) - K, 0) \;=\; \max(\$100 - \$100, 0) \;=\; \$0$$

$$C_3(t+2) \;=\; \max(S_3(t+2) - K, 0) \;=\; \max(\$44.4\bar{4} - \$100, 0) \;=\; \$0$$



Call option prices

We know that $C_2(t+1) = 0$ since, if $S(t+1) = S_2(t+1)$, then no matter what happens in the next step, the call option will not be exercised.

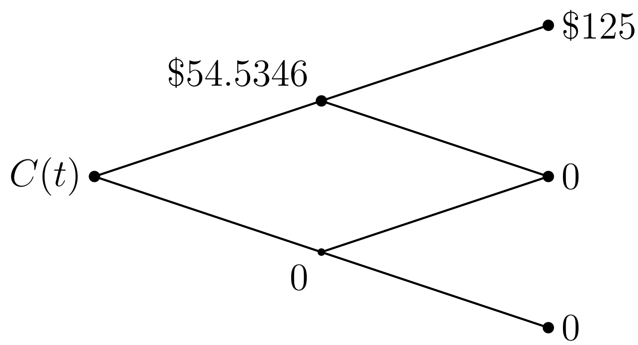
To find out $C_1(t+1)$, we use the formula:

$$C_1(t+1) \;=\; (1+r\Delta)^{-1} E^{\tilde{P}}[C(t+2) \,|\, S(t+1) = S_1(t+1)]\,.$$

We mainly write the conditional probability for our own edification.

It just reminds us that we do not have to include the probability that $S(t+1)$ actually equals $S_1(t+1)$, just the two probabilities for the next step.

$$\begin{aligned} C_1(t+1) &= \frac{\tilde{P}_{11} \cdot C_1(t+2) + \tilde{P}_{12} \cdot C_2}{1.0475} \\ &= \frac{(0.457)\$125 + (0.543)0}{1.0475} \\ &= \frac{\$57.125}{1.0475} \\ &\approx \$54.5346 \end{aligned}$$



Call option prices

Finally, we use the formula

$$C(t) \;=\; (1+r\Delta)^{-1}\, E^{\tilde{P}}[C(t+1)]\,.$$

(Now we do include the probabilities for the first step, and only those.)

$$\begin{aligned} C(t) &= \frac{\tilde{P}_1 \cdot C_1(t+1) + \tilde{P}_2 \cdot C_2}{1.0475} \\ &= \frac{(0.457)\$54.5346 + (0.543)0}{1.0475} \\ &= \frac{\$24.9223}{1.0475} \\ &\approx \$23.7922 \\ &\approx \$23.79 \end{aligned}$$

Perspective

At this point, we have covered the highlights of what would typically be covered in about the first 1/3 *of an undergraduate math-finance class.*

- *Introduced "derivatives"*
- *Talked about arbitrage and the no-arbitrage condition*
- *Priced a "fair forward contract"*
- *Calculated payoffs at expiration for call options and put options*
- *Discussed put-call parity*
- *Priced a call option in the binomial tree model*

What we have left out, that Hull would have covered, are all the other derivatives, such as interest rate derivatives. Also, we have avoided economic discussions, as to why stock prices are what they are, etc. So we covered "Hull-lite". That is for a reason. We are now going to enter into the real model for stock prices, which involves Brownian motion, also known as the Wiener process. This is heavy-duty mathematics, combining both real analysis and probability and stochastic calculus. For this reason, right now the perspective of the class is going to change noticably to focus more on math. At first we will review some things from advanced calculus and probability. But ultimately we will be talking about completely new topics such as: random walk, Brownian motion, and stochastic (Itô) integrals.

For Friday start reading Chapter 3.