Lecture Notes for Math 210 – 17 September 2007

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1 Chapter 2: A Primer on the Arbitrage Theorem

Subtitle: The binomial tree model

Theorem

In the binomial model, the principle of *no-arbitrage* implies that there are two constants $\psi_1, \psi_2 \geq 0$ such that

$$\begin{bmatrix} B(t) \\ S(t) \\ C(t) \end{bmatrix} = \begin{bmatrix} B(t+\Delta) & B(t+\Delta) \\ S_1(t+\Delta) & S_2(t+\Delta) \\ C_1(t+\Delta) & C_2(t+\Delta) \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \qquad (*)$$

Assuming $S_1(t+\Delta) \neq S_2(t+\Delta)$, the constants ψ_1 and ψ_2 are unique.

1.1 Proof of the theorem

Suppose $S_1(T) > S_2(T)$.

Then last time we found ψ_1 and ψ_2 such that

$$\begin{bmatrix} B(t) \\ S(t) \end{bmatrix} = \begin{bmatrix} B(T) & B(T) \\ S_1(T) & S_2(T) \end{bmatrix} \cdot \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \qquad (**)$$

We used Cramer's rule for inverting 2×2 matrices.

But the answer is

$$\psi_1 = \frac{1}{1+r\Delta} \cdot \frac{(1+r\Delta)S(t) - S_2(T)}{S_1(T) - S_2(T)} \quad \text{and} \quad \psi_2 = \frac{1}{1+r\Delta} \cdot \frac{S_1(T) - (1+r\Delta)S(t)}{S_1(T) - S_2(T)}.$$

The condition for ψ_1 and ψ_2 to be nonnegative is that

$$\frac{S_2(T)}{1+r\Delta} \le S(t) \le \frac{S_1(T)}{1+r\Delta}.$$

Q: What is $S(t)$? What are the interpretations of $\frac{S_1(T)}{1+r\Delta}$ and $\frac{S_2(T)}{1+r\Delta}$?

A: $S(t)$ is the spot price for the stock at time t, today. $S_1(T)$ and $S_2(T)$ are the maximum and minimum value that the stock can take at time T. So $\frac{S_1(T)}{1+r\Delta}$ and $\frac{S_2(T)}{1+r\Delta}$ are the maximum and minimum present values for the stock (no matter which the stock actually takes).

If we know that the stock will be at least $S_2(T)$ at time T in the future. Then its payoff is at least that of a bond whose payoff is $S_2(T)$ in the future. Therefore, the stock's present value is at least the value of that bond, which is $S_2(T)/(1+r\Delta)$.

Since we know (∗∗), and we know that ψ_1 and ψ_2 are nonnegative, all that remains is to prove that

$$C(t) = \begin{bmatrix} C_1(T) & C_2(T) \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} .$$

But note that there are also constants ϕ_1 and ϕ_2 such that

$$\begin{bmatrix} C_1(T) & C_2(T) \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 \end{bmatrix} \begin{bmatrix} B(T) & B(T) \\ S_1(T) & S_2(T) \end{bmatrix}$$

Namely, take

$$\begin{bmatrix} \phi_1 & \phi_2 \end{bmatrix} = \begin{bmatrix} C_1(T) & C_2(T) \end{bmatrix} \begin{bmatrix} B(T) & B(T) \\ S_1(T) & S_2(T) \end{bmatrix}^{-1} .$$

This works because, generally, if we want to solve the equation

$$b^T = x^T A$$

then taking

$$x^T = b^T A^{-1} ,$$

we have

$$x^T A = (b^T A^{-1})A = b^T(A^{-1}A) = b^T I = b^T ,$$

which is what we wanted.

Consider the following portfolios:

- **Portfolio 1:** Long 1 call option.
- **Portfolio 2:** ϕ_1 times a bond, and ϕ_2 times the stock.
  - If either of ϕ_1 or ϕ_2 is negative, that is okay, it just means you short-sell instead of buy.
  - We also assume you can buy or sell fractions of the stock and bond.
  - But that is okay, since we can multiply both portfolios by a large positive integer to make all the numbers (approximately) integer.

Both portfolios have the same value at time T.

Fact (replicating portfolios): Suppose two portfolios have the same value at time T. Then they must have the same value today. Otherwise, we could take the long position on the cheaper one, and the short position on the costlier one. At time T we will have made a risk-free profit.

(I think of this by the saying "My money is as green as yours." If I own Portfolio 1 and it is guaranteed to have the same value at time T as Portfolio 2, then there is no reason for anybody to give me less today for my portfolio then for the other one.)

By the principle of *replicating portfolios*, we know that the present value of Portfolio 1 must equal the present value of Portfolio 2.

So,

$$C(t) \;=\; \phi_1 \cdot B(t) + \phi_2 \cdot S(t)\,.$$

This can be re-written using matrix multiplication:

$$C(t) \;=\; \begin{bmatrix} \phi_1 & \phi_2 \end{bmatrix} \begin{bmatrix} B(t) \\ S(t) \end{bmatrix}$$

But, using equation $(**)$, we then have

$$\begin{aligned} C(t) \;&=\; \begin{bmatrix} \phi_1 & \phi_2 \end{bmatrix} \begin{bmatrix} B(T) & B(T) \\ S_1(T) & S_2(T) \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \\ &=\; \begin{bmatrix} C_1(T) & C_2(T) \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \end{aligned}$$

That is all we needed to check, so that finishes the proof.

$\square$

1.2 Synthetic probabilities

Define

$$\tilde{P}_1 = (1 + r\Delta)\psi_1 = \frac{(1 + r\Delta)S(t) - S_2(T)}{S_1(T) - S_2(T)}$$

and

$$\tilde{P}_2 = (1 + r\Delta)\psi_2 = \frac{S_1(T) - (1 + r\Delta)S(t)}{S_1(T) - S_2(T)}$$

Since ψ_1 and ψ_2 are nonnegative, the same is true of $\tilde{P}_1$ and $\tilde{P}_2$.

Also

$$\begin{aligned} \tilde{P}_1 + \tilde{P}_2 &= \frac{(1 + r\Delta)S(t) - S_2(T)}{S_1(T) - S_2(T)} + \frac{S_1(T) - (1 + r\Delta)S(t)}{S_1(T) - S_2(T)} \\ &= \frac{S_1(T) - S_2(T)}{S_1(T) - S_2(T)} \\ &= 1\,. \end{aligned}$$

Recall "probabilities": A set of numbers assigned to possible outcomes of an unknown random variable are called *probabilities* if they are all nonnegative and add up to 1.

So $\tilde{P}_1$ and $\tilde{P}_2$ are probabilities for the two possible outcomes of $S(t + \Delta)$: namely, $S_1(t + \Delta)$ and $S_2(t + \Delta)$.

$\star$ Note: The probabilities $\tilde{P}_1$ and $\tilde{P}_2$ are called the "synthetic probabilities" or "risk-neutral" probabilities. These are *not* the actual probabilities for the stock to go up or down. So far we have not talked about those probabilities, and we haven't needed them. We will discuss all this briefly in a few minutes.

Recall "expectation": If a random variable X can take values $x_1, \ldots, x_n$, with probabilities $p_1, \ldots, p_n$, then we say the *expectation* of X is

$$E^p[X] \;:=\; p_1 \cdot x_1 + \cdots + p_n \cdot c_n \,.$$

So,

$$E^{\tilde{P}}[S(t+\Delta)] \;:=\; \tilde{P}_1 S_1(t+\Delta) + \tilde{P}_2 S_2(t+\Delta)\,,$$

and

$$E^{\tilde{P}}[C(t+\Delta)] \;:=\; \tilde{P}_1 C_1(t+\Delta) + \tilde{P}_2 C_2(t+\Delta)\,.$$

We write a colon-equal := to denote that these are definitions, not results of calculations.

But if we do calculate, note what happens:

$$\begin{aligned} E^{\tilde{P}}[S(T)] \;&:=\; \tilde{P}_1 S_1(T) + \tilde{P}_2 S_2(T) \\ &=\; (1+r\Delta)\psi_1 S_1(T) + (1+r\Delta)\psi_2 S_2(T) \\ &=\; (1+r\Delta)\begin{bmatrix} S_1(T) & S_2(T) \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \\ &=\; (1+r\Delta) S(t)\,, \end{aligned}$$

and similarly

$$E^{\tilde{P}}[C(T)] \;=\; (1+r\Delta) C(t)\,.$$

Thus,

1. $\tilde{P}_1$ and $\tilde{P}_2$ are the *unique* probabilities such that $S(t)$ equals the present value of $E^{\tilde{P}}[S(T)]$. In other words, that is how $\tilde{P}_1$ and $\tilde{P}_2$ are calculated.

2. Using these probabilities, we can calculate $C(t)$ as the present value of $E^{\tilde{P}}[C(T)]$.

This interpretation is simple, but important.

For the purpose of calculating $C(t)$ we can summarize as follows:

- First, calculate $\tilde{P}_1$ and $\tilde{P}_2$ as

$$\tilde{P}_1 = \frac{(1+r\Delta)S(t) - S_2(T)}{S_1(T) - S_2(T)} \quad \text{and} \quad \tilde{P}_2 = \frac{S_1(T) - (1+r\Delta)S(t)}{S_1(T) - S_2(T)}$$

- Second, calculate $C_1(T)$ and $C_2(T)$ using the formulas

$$C_1(T) = \max(S_1(T) - K, 0) \quad \text{and} \quad C_2(T) = \max(S_2(T) - K, 0)\,.$$

- Third, calculate $E^{\tilde{P}}[C(T)]$ as

$$E^{\tilde{P}}[C(T)] = \tilde{P}_1 \cdot C_1(T) + \tilde{P}_2 \cdot C_2(T)\,.$$

- Finally, set $C(t) = (1+r\Delta)^{-1}\, E^{\tilde{P}}[C(T)]$.

<u>*Difference between real probabilities and synthetic probabilities:*</u> *Compared to synthetic probabilities, investors must be paid a premium for taking a risk. Real probabilities would result in a higher expectation than* $(1+r\Delta)S(t)$*. Otherwise, sound investors would choose to just invest in a bond. But our underlying argument for*

$C(t)$ relied on the no-arbitrage principle, not the probabilities themselves. So $C(t)$ is the correct price, and the synthetic probabilities are just a convenient way to remember the formulas coming out of the no-arbitrage theorem.

Next time: Two-step binomial trees