# Lecture Notes for Math 210 – 12 September 2007

Shannon Starr

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## 1 Chapter 2: A Primer on the Arbitrage Theorem

### Subtitle: The binomial tree model

**Theorem**

In the binomial model, the principle of *no-arbitrage* implies that there are two constants $\psi_1, \psi_2 \geq 0$ such that

$$\begin{bmatrix} B(t) \\ S(t) \\ C(t) \end{bmatrix} = \begin{bmatrix} B(t+\Delta) & B(t+\Delta) \\ S_1(t+\Delta) & S_2(t+\Delta) \\ C_1(t+\Delta) & C_2(t+\Delta) \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \qquad (*)$$

Assuming $S_1(t+\Delta) \neq S_2(t+\Delta)$, the constants $\psi_1$ and $\psi_2$ are unique.

**Recall:** We know $B(t)$ and $S(t)$ from the stock/bond listings.

We can calculate $B(t + \Delta) = (1 + r\Delta)B(t)$.

We assume we know the numbers $S_1(t + \Delta)$ and $S_2(t + \Delta)$.

We can calculate $C_1(t + \Delta)$ and $C_2(t + \Delta)$. We do not know $\psi_1$, $\psi_2$ or $C(t)$, yet.

<u>Idea:</u> Solve for $\psi_1$ and $\psi_2$, first, and then for $C(t)$, using equation $(*)$.

### 1.1 Example

Suppose a stock has price $S(t) = \$100$, today.

Suppose we want to know the price for a call option:

- $T = t + \Delta$ with $\Delta = 3$ months $= 0.25$ years.

- $K = \$100$ (at the money)

Suppose (hypothetically) that we know that $S(t+\Delta)$ will be one of the two values:

$$S(t + \Delta) = \begin{cases} \$110.00 & \text{if stock goes up,} \\ \\ \$95.00 & \text{if stock goes down.} \end{cases}$$

Suppose we know the borrow/lend/bond rate for 3 months is $r = 5\%$.

What should $C(t)$ be?

**Answer.** Step 1: Set-up.

Let us write $B(t) = \$100$, just to have something to put in.

Then

$$\begin{aligned} B(t+\Delta) \;=\; B(t+0.25) \;&=\; \$100(1+r\Delta) \\ &=\; \$100(1+(0.05)(0.25)) \\ &=\; \$101.25\,. \end{aligned}$$

Using equation $(*)$,

$$\begin{bmatrix} \$100.00 \\ \$100 \\ C(t) \end{bmatrix} = \begin{bmatrix} \$101.25 & \$101.25 \\ \$110 & \$95 \\ \$10 & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$

(Reminder: matrix multiplication, across the row for first matrix and down the column for second.)

$$\begin{bmatrix} \$100.00 \\ \$100 \\ C(t) \end{bmatrix} = \begin{bmatrix} (\$101.25)\psi_1 + (\$101.25)\psi_2 \\ (\$110)\psi_1 + (\$95)\psi_2 \\ (\$10)\psi_1 + 0\cdot\psi_2 \end{bmatrix} \qquad (**)$$

Step 2: Use the first two rows of $(**)$ to solve for $\psi_1$ and $\psi_2$

$$\begin{aligned} \$100.00 \;&=\; (\$101.25)\psi_1 + (\$101.25)\psi_2 \\ \$100 \;&=\; (\$110)\psi_1 + (\$95)\psi_2 \end{aligned}$$

From first equation

$$\frac{100}{101.25} = \psi_1 + \psi_2 \qquad \Longrightarrow \qquad \psi_2 = -\psi_1 + \frac{100}{101.50}$$
$$= -\psi_1 + 0.98765$$

Substituting this into the second equation

$$\begin{aligned}
\$100 &= (\$110)\psi_1 + (\$95)\psi_2 \\
&= (\$110)\psi_1 + (\$95)\left(-\psi_1 + 0.98765\right) \\
&= (\$110 - \$95)\psi_1 + \$93.827 \\
&\Rightarrow \quad (\$15)\psi_1 = \$100 - \$93.827 = \$6.173 \\
&\Rightarrow \quad \psi_1 = \frac{6.173}{15} = 0.4115
\end{aligned}$$

and plugging back into the equation for $\psi_2$,

$$\begin{aligned}
\psi_2 &= -\psi_1 + 0.98765 \\
&= -0.4115 + 0.98765 = 0.5761
\end{aligned}$$

Step 3: Use $\psi_1$, $\psi_2$ and the third row of $(*)$ to solve for $C(t)$.

$$C(t) \;=\; (\$10)\psi_1 + 0\cdot\psi_2 \;=\; (\$10)(0.4115) + 0\cdot(0.5761) \;=\; \$4.12\,.$$

So $C(t) = \$4.12$.

This is the first time we have been able to calculate $C(t)$.

Of course, it is for a very simplified model, which assumes the unrealistic possibility that we know the only two possible values for $S(t+\Delta)$.

### 1.2 Proof of the theorem

We want to prove that there are nonnegative constants $\psi_1$ and $\psi_2$ such that

$$\begin{bmatrix} B(t) \\ S(t) \\ C(t) \end{bmatrix} = \begin{bmatrix} B(T) & B(T) \\ S_1(T) & S_2(T) \\ C_1(T) & C_2(T) \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$

Suppose $S_1(T) > S_2(T)$.

Then the matrix

$$\begin{bmatrix} B(T) & B(T) \\ S_1(T) & S_2(T) \end{bmatrix}$$

is invertible.

By Cramer's rule,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

as long as $ad - bc \neq 0$. Applying this to our matrix, we get

$$\begin{bmatrix} B(T) & B(T) \\ S_1(T) & S_2(T) \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{S_2(T)}{B(T)\delta} & \frac{1}{\delta} \\ \frac{S_1(T)}{B(T)\delta} & -\frac{1}{\delta} \end{bmatrix},$$

where

$$\delta = S_1(T) - S_2(T).$$

We can find some constants $\psi_1$ and $\psi_2$ (we hold off on nonnegativity) such that

$$\begin{bmatrix} B(t) \\ S(t) \end{bmatrix} = \begin{bmatrix} B(T) & B(T) \\ S_1(T) & S_2(T) \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \qquad (***)$$

by the formula

$$\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} B(T) & B(T) \\ S_1(T) & S_2(T) \end{bmatrix}^{-1} \begin{bmatrix} B(t) \\ S(t) \end{bmatrix}$$

Now we also want to prove that

$$C(t) = \begin{bmatrix} C_1(T) & C_2(T) \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}.$$

Aside from proving that $\psi_1$ and $\psi_2$ are nonnegative, that is the only thing left for proving the theorem.

But note that there are also constants $\phi_1$ and $\phi_2$ such that

$$\begin{bmatrix} C_1(T) & C_2(T) \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 \end{bmatrix} \begin{bmatrix} B(T) & B(T) \\ S_1(T) & S_2(T) \end{bmatrix}$$

Namely, take

$$\begin{bmatrix} \phi_1 & \phi_2 \end{bmatrix} = \begin{bmatrix} C_1(T) & C_2(T) \end{bmatrix} \begin{bmatrix} B(T) & B(T) \\ S_1(T) & S_2(T) \end{bmatrix}^{-1}.$$

Consider the following portfolios:

- **Portfolio 1:** Long 1 call option.
- **Portfolio 2:** $\phi_1$ times a bond, and $\phi_2$ times the stock.
  - If either of $\phi_1$ or $\phi_2$ is negative, that is okay, it just means you short-sell instead of buy.
  - We also assume you can buy or sell fractions of the stock and bond.
  - But that is okay, since we can multiply both portfolios by a large positive integer to make all the numbers (approximately) integer.

Since both portfolios have the same value at time $T$, they must both have the same value now. So,

$$\begin{aligned} C(t) &= \phi_1 \cdot B(t) + \phi_2 \cdot S(t) \\ &= \begin{bmatrix} \phi_1 & \phi_2 \end{bmatrix} \begin{bmatrix} B(t) \\ S(t) \end{bmatrix} \end{aligned}$$

But, using equation $(***)$, we then have

$$C(t) = \begin{bmatrix} \phi_1 & \phi_2 \end{bmatrix} \begin{bmatrix} B(T) & B(T) \\ S_1(T) & S_2(T) \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$

$$= \begin{bmatrix} C_1(T) & C_2(T) \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} .$$

Finally, let us check that $\psi_1$ and $\psi_2$ are nonnegative.

Recall that

$$\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} B(T) & B(T) \\ S_1(T) & S_2(T) \end{bmatrix}^{-1} \begin{bmatrix} B(t) \\ S(t) \end{bmatrix}$$

But we also had a formula for the matrix inverse:

$$\begin{bmatrix} B(T) & B(T) \\ S_1(T) & S_2(T) \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{S_2(T)}{B(T)\delta} & \frac{1}{\delta} \\ \frac{S_1(T)}{B(T)\delta} & -\frac{1}{\delta} \end{bmatrix}$$

Also, remember that $B(T) = (1 + r\Delta)B(t)$.

Using all this, we get

$$\psi_1 = \frac{(1 + r\Delta)S(t) - S_2(T)}{(1 + r\Delta)\delta} \quad \text{and} \quad \psi_2 = \frac{S_1(T) - (1 + r\Delta)S(t)}{(1 + r\Delta)\delta} .$$

Since $\delta > 0$, this means $\psi_1$ and $\psi_2$ are both nonnegative if and only if

$$S_2(T) \le (1 + r\Delta)S(t) \le S_1(T) ,$$

or in other words

$$\frac{S_2(T)}{1+r\Delta} \;\le\; S(t) \;\le\; \frac{S_1(T)}{1+r\Delta}\,.$$

Since we know that, for sure, the future value of the stock is at least $S_1(T)$ and at most $S_2(T)$, these inequalities also follow by the no-arbitrage condition.

## 1.3 Synthetic probabilities

Define

$$\tilde{P}_1 \;=\; (1+r\Delta)\psi_1 \;=\; \frac{(1+r\Delta)S(t) - S_2(T)}{\delta}$$

and

$$\tilde{P}_2 \;=\; (1+r\Delta)\psi_2 \;=\; \frac{S_1(T) - (1+r\Delta)S(t)}{\delta}\,.$$

Since $\psi_1$ and $\psi_2$ are nonnegative, the same is true of $\tilde{P}_1$ and $\tilde{P}_2$. Also

$$\begin{aligned}
\tilde{P}_1 + \tilde{P}_2 \;&=\; \frac{(1+r\Delta)S(t) - S_2(T)}{\delta} + \frac{S_1(T) - (1+r\Delta)S(t)}{\delta} \\
&=\; \frac{S_1(T) - S_2(T)}{\delta} \\
&=\; \frac{S_1(T) - S_2(T)}{S_1(T) - S_2(T)} \\
&=\; 1\,,
\end{aligned}$$

because, remember that $\delta = S_1(T) - S_2(T)$.

<u>Probabilities</u> A set of numbers assigned to possible outcomes of an unknown random variable are called *probabilities* if they are all nonnegative and add up to 1. So $\tilde{P}_1$ and $\tilde{P}_2$ are probabilities for the two possible outcomes $S(t+\Delta)$: namely, $S_1(t+\Delta)$ and $S_2(t+\Delta)$.

<u>Expectation</u> If a random variable $X$ can take values $x_1, \ldots, x_n$, with probabilities $p_1, \ldots, p_n$, then we say the *expectation* of $X$ is

$$E^p[X] := p_1 \cdot x_1 + \cdots + p_n \cdot c_n \,.$$

So, for example,

$$E^{\tilde{P}}[S(t+\Delta)] := \tilde{P}_1 S_1(t+\Delta) + \tilde{P}_2 S_2(t+\Delta) \,,$$

and

$$E^{\tilde{P}}[C(t+\Delta)] := \tilde{P}_1 C_1(t+\Delta) + \tilde{P}_2 C_2(t+\Delta) \,.$$

We write a colon-equal := to denote that these are definitions, not results of calculations.

But if we do calculate, note what happens:

$$\begin{aligned} E^{\tilde{P}}[S(T)] &:= \tilde{P}_1 S_1(T) + \tilde{P}_2 S_2(T) \\ &= (1+r\Delta)\psi_1 S_1(T) + (1+r\Delta)\psi_2 S_2(T) \\ &= (1+r\Delta) \begin{bmatrix} S_1(T) & S_2(T) \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \\ &= (1+r\Delta) S(t) \,, \end{aligned}$$

and similarly

$$E^{\tilde{P}}[C(T)] = (1+r\Delta) C(t) \,.$$

Thus, $\tilde{P}_1$ and $\tilde{P}_2$ are the *unique* probabilities such that

$$S(t) \;=\; \text{present value of } E^{\tilde{P}}[S(T)]\,.$$

Using these probabilities, we can calculate $C(t)$ because

$$C(t) \;=\; \text{present value of } E^{\tilde{P}}[C(T)]\,.$$

This interpretation is simple, but important.