Lecture Notes for Math 210 – 12 September 2007

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1 Chapter 2: A Primer on the Arbitrage Theorem

Subtitle: The binomial tree model

Notation: t will represent time. We now write a stock price as $S(t)$ instead of S_t.

$B(t)$ = price (or value) of a bond at time t.

$S(t)$ = price of a stock at time t.

$C(t)$ = price of a call option at time t.

We sometimes still mean t to mean the time today.

Q: If t = date (and time) today, which of $B(t)$, $S(t)$ and $C(t)$ do we know?

Answer: Current stock and bond prices are listed in newspaper and online.

As financial mathematicians, it is our job to find $C(t)$.

1.1 Arbitrage theorem for the binomial model

Suppose at time t we know the price of $B(t)$ and $S(t)$.

Suppose the call has expiration date $T = t + \Delta$, where Δ is "small".

(E.g., $\Delta \leq 1$ year.)

- If r = borrowing/lending interest rate, then

$$B(t + \Delta) \;=\; (1 + r\Delta)\, B(t)\,.$$

- *Binomial model for stock price.*

  We do not know $S(t + \Delta)$, today.

  - But suppose that we do know 2 numbers, $S_1(t + \Delta)$ and $S_2(t + \Delta)$, and we know that $S(t + \Delta)$ will be one of these two.
  - The only thing we don't know is which of these two it will take.
  - This is a purely hypothetical model.

- Then the payoff at expiration must take one of exactly two possible values:

  - if $S(T) = S_1(T)$, then

$$C_1(T) \;=\; \max(S_1(T) - K, 0)\,,$$

  - if $S(T) = S_2(T)$, then

$$C_2(T) \;=\; \max(S_2(T) - K, 0)\,.$$

Theorem

Suppose $S_1(t+\Delta)$ and $S_2(t+\Delta)$ both have positive probability to occur, and no other outcomes for $S(t+\Delta)$ are possible. Then the principle of *no-arbitrage* implies that there are two constants $\psi_1, \psi_2 \geq 0$ such that

$$\begin{bmatrix} B(t) \\ S(t) \\ C(t) \end{bmatrix} = \begin{bmatrix} B(t+\Delta) & B(t+\Delta) \\ S_1(t+\Delta) & S_2(t+\Delta) \\ C_1(t+\Delta) & C_2(t+\Delta) \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \qquad (*)$$

Assuming $S_1(t+\Delta) \neq S_2(t+\Delta)$, the constants ψ_1 and ψ_2 are unique.

Q: In equation $(*)$ what is known, and what is unknown?

Answer: We know $B(t)$ and $S(t)$ from the stock/bond listings.

We can calculate $B(t+\Delta)$ because $B(t+\Delta) = (1+r\Delta)B(t)$.

We assume we know the numbers $S_1(t+\Delta)$ and $S_2(t+\Delta)$.

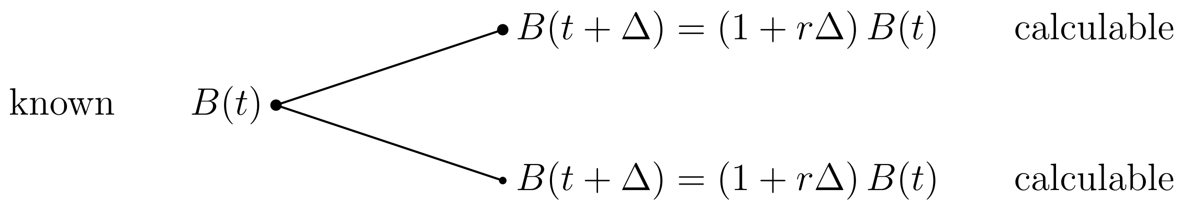
We can calculate $C_1(t+\Delta)$ and $C_2(t+\Delta)$:

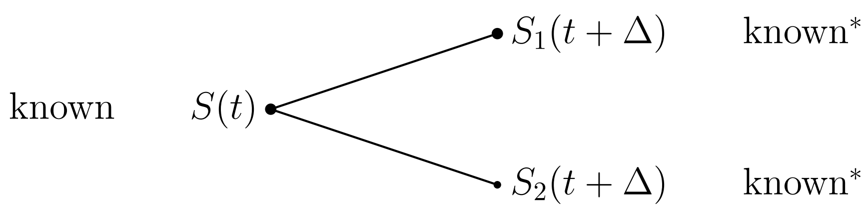
$$\begin{aligned} C_1(t+\Delta) &= \max(S_1(t+\Delta) - K, 0)\,, \\ C_2(t+\Delta) &= \max(S_2(t+\Delta) - K, 0)\,. \end{aligned}$$

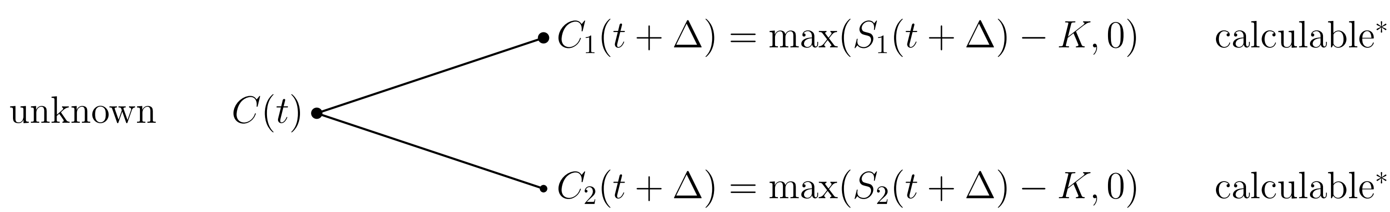
We do not know what $C(t)$ is, yet.

We do not know what ψ_1 and ψ_2 are, yet.

But we can solve for ψ_1 and ψ_2, first, and then for $C(t)$, using equation $(*)$.


known
$B(t)$
$B(t+\Delta) = (1 + r\Delta)\, B(t)$
calculable
$B(t+\Delta) = (1 + r\Delta)\, B(t)$
calculable


known
$S(t)$
$S_1(t+\Delta)$
known*
$S_2(t+\Delta)$
known*


unknown
$C(t)$
$C_1(t+\Delta) = \max(S_1(t+\Delta) - K, 0)$
calculable*
$C_2(t+\Delta) = \max(S_2(t+\Delta) - K, 0)$
calculable*

1.2 Example

Suppose a stock has price $S(t) = \$100$, today.

Suppose we want to know the price for a call option:

- $T = t + \Delta$ with $\Delta = 3$ months $= 0.25$ years.
- $K = \$100$ (at the money)

Suppose (hypothetically) that we know that $S(t+\Delta)$ will be one of the two values:

$$S(t+\Delta) \;=\; \begin{cases} \$110.00 & \text{if stock goes up,} \\ \$95.00 & \text{if stock goes down.} \end{cases}$$

Suppose we know the borrow/lend/bond rate for 3 months is $r = 5\%$.

What should $C(t)$ be?

Answer. Step 1: Set-up.

Let us write $B(t) = \$100$, just to have something to put in.

Then

$$\begin{aligned} B(t+\Delta) \;=\; B(t+0.25) \;&=\; \$100(1 + r\Delta) \\ &=\; \$100(1 + (0.05)(0.25)) \\ &=\; \$101.25\,. \end{aligned}$$

Using equation (∗),

$$\begin{bmatrix} \$100.00 \\ \$100 \\ C(t) \end{bmatrix} = \begin{bmatrix} \$101.25 & \$101.25 \\ \$110 & \$95 \\ \$10 & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$

(Reminder: matrix multiplication, across the row for first matrix and down the column for second.)

$$\begin{bmatrix} \$100.00 \\ \$100 \\ C(t) \end{bmatrix} = \begin{bmatrix} (\$101.25)\psi_1 + (\$101.25)\psi_2 \\ (\$110)\psi_1 + (\$95)\psi_2 \\ (\$10)\psi_1 + 0 \cdot \psi_2 \end{bmatrix}$$

<u>Step 2:</u> Use the first two rows of (∗) to solve for ψ_1 and ψ_2

$$\begin{aligned} \$100.00 &= (\$101.25)\psi_1 + (\$101.25)\psi_2 \\ \$100 &= (\$110)\psi_1 + (\$95)\psi_2 \end{aligned}$$

From first equation

$$\frac{100}{101.25} = \psi_1 + \psi_2 \qquad \Longrightarrow \qquad \begin{aligned} \psi_2 &= -\psi_1 + \frac{100}{101.50} \\ &= -\psi_1 + 0.98765 \end{aligned}$$

Substituting this into the second equation

$$\begin{aligned}
\$100 &= (\$110)\psi_1 + (\$95)\,(-\psi_1 + 0.98765) \\
&= (\$110 - \$95)\psi_1 - \$93.827 \\
&\Rightarrow \quad (\$15)\psi_1 \;=\; \$100 - \$93.827 \;=\; \$6.173 \\
&\Rightarrow \quad \psi_1 \;=\; \frac{6.173}{15} \;=\; 0.4115
\end{aligned}$$

and plugging back into the equation for ψ_2,

$$\psi_2 \;=\; -0.4115 + 0.98765 \;=\; 0.5761$$

Step 3: Use ψ_1, ψ_2 and the third row of $(*)$ to solve for $C(t)$.

$$C(t) \;=\; (\$10)\psi_1 + 0 \cdot \psi_2 \;=\; (\$10)(0.4115) + 0 \cdot (0.5761) \;=\; \$4.12\,.$$

1.3 Proof of the theorem

We want to prove that there are nonnegative constants ψ_1 and ψ_2 such that

$$\begin{bmatrix} B(t) \\ S(t) \\ C(t) \end{bmatrix} = \begin{bmatrix} B(T) & B(T) \\ S_1(T) & S_2(T) \\ C_1(T) & C_2(T) \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$

Suppose $S_1(T) > S_2(T)$.

Then the matrix

$$\begin{bmatrix} B(T) & B(T) \\ S_1(T) & S_2(T) \end{bmatrix}$$

is invertible.

By Cramer's rule,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

as long as $ad - bc \neq 0$. Applying this to our matrix, we get

$$\begin{bmatrix} B(T) & B(T) \\ S_1(T) & S_2(T) \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{S_2(T)}{B(T)\delta} & \frac{1}{\delta} \\ \frac{S_1(T)}{B(T)\delta} & -\frac{1}{\delta} \end{bmatrix},$$

where

$$\delta = S_1(T) - S_2(T).$$

We can find some constants ψ_1 and ψ_2 (we hold off on nonnegativity) such that

$$\begin{bmatrix} B(t) \\ S(t) \end{bmatrix} = \begin{bmatrix} B(T) & B(T) \\ S_1(T) & S_2(T) \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$

by the formula

$$\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} B(T) & B(T) \\ S_1(T) & S_2(T) \end{bmatrix}^{-1} \begin{bmatrix} B(t) \\ S(t) \end{bmatrix}$$

Now we also want to prove that

$$C(t) = \begin{bmatrix} C_1(T) & C_2(T) \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} .$$

Aside from proving that ψ_1 and ψ_2 are nonnegative, that is the only thing left for proving the theorem.

But note that there are also constants ϕ_1 and ϕ_2 such that

$$\begin{bmatrix} C_1(T) & C_2(T) \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 \end{bmatrix} \begin{bmatrix} B(T) & B(T) \\ S_1(T) & S_2(T) \end{bmatrix}$$

Namely, take

$$\begin{bmatrix} \phi_1 & \phi_2 \end{bmatrix} = \begin{bmatrix} C_1(T) & C_2(T) \end{bmatrix} \begin{bmatrix} B(T) & B(T) \\ S_1(T) & S_2(T) \end{bmatrix}^{-1} .$$

Consider the following portfolios:

- **Portfolio 1:** Long 1 call option.

- **Portfolio 2:** ϕ_1 times a bond, and ϕ_2 times the stock.

If either of ϕ_1 or ϕ_2 is negative, that is okay, it just means you short-sell instead of buy.

We also assume you can buy or sell fractions of the stock and bond.

But that is okay, since we can multiply both portfolios by a large integer to make all the numbers (nearly) integer.

Since both portfolios have the same value at time T, they must both have the same value now. So,

$$\begin{aligned} C(t) &= \begin{bmatrix} \phi_1 & \phi_2 \end{bmatrix} \begin{bmatrix} B(t) \\ S(t) \end{bmatrix} \\ &= \begin{bmatrix} \phi_1 & \phi_2 \end{bmatrix} \begin{bmatrix} B(T) & B(T) \\ S_1(T) & S_2(T) \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \\ &= \begin{bmatrix} C_1(T) & C_2(T) \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}. \end{aligned}$$

Finally, let us check that ψ_1 and ψ_2 are nonnegative.

(Note that it doesn't matter if ϕ_1 or ϕ_2 are negative since we can short-sell, if necessary, to make the replicating portfolio.)

We use the formulas from before

$$\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} B(T) & B(T) \\ S_1(T) & S_2(T) \end{bmatrix}^{-1} \begin{bmatrix} B(t) \\ S(t) \end{bmatrix}$$

Remember that we had a formula for the matrix inverse:

$$\begin{bmatrix} B(T) & B(T) \\ S_1(T) & S_2(T) \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{S_2(T)}{B(T)\delta} & \frac{1}{\delta} \\ \frac{S_1(T)}{B(T)\delta} & -\frac{1}{\delta} \end{bmatrix}$$

Also, remember that $B(T) = (1 + r\Delta)B(t)$.

Using all this, we get

$$\psi_1 = \frac{(1 + r\Delta)S(t) - S_2(T)}{(1 + r\Delta)\delta} \quad \text{and} \quad \psi_2 = \frac{S_1(T) - (1 + r\Delta)S(t)}{(1 + r\Delta)\delta}.$$

Since $\delta > 0$, this means ψ_1 and ψ_2 are both nonnegative if and only if

$$S_2(T) \le (1 + r\Delta)S(t) \le S_1(T),$$

or in other words

$$\frac{S_2(T)}{1 + r\Delta} \le S(t) \le \frac{S_1(T)}{1 + r\Delta}.$$

Since we know that, for sure, the future value of the stock is at least $S_1(T)$ and at most $S_2(T)$, these inequalities also follow by the no-arbitrage condition.